Classification of Adinkra Graphs

A Senior Project submitted to
The Division of Science, Mathematics, and Computing
of
Bard College

by
Sylvia Naples

Annandale-on-Hudson, New York
May, 2009

Abstract

In 2004, Faux and Gates introduced Adinkras as a tool for studying supersymmetry. An Adinkra is a graph (a set of vertices and edges connecting those vertices) with some additional properties. Adinkras are bipartite and edge-N-regular with N edge colors. The bipartition corresponds to two classes of fundamental particles, bosons and fermions. The edges connecting those vertices represent supersymmetric transformations between the particles.

The vertices of an Adinkra are given a height assignment. Up to isomorphism, all Adinkras with the same topology are related by a vertex lowering operation, which allows sources in the graph to be moved to lower heights. Our interest is in classifying the Adinkras with respect to the given height assignment. We compute an invariant on the graphs: a sequence of natural numbers called Betti numbers given by a process called homology. Our main result is a proof that these sequences are given by polynomials with degree less than $N-1$. We generate a spanning set for the polynomials and construct a single equation that gives the sequence of Betti numbers for any Adinkra. The equation depends only on N, the number of vertices at each height, and the number of sources at each height.

Contents

Dedication

For Chris Tarsell and Jennifer Day

Acknowledgments

First, I would like to thank my advisor, Greg Landweber, for his time and enthusiasm on this project, and the members of my board, Rebecca Thomas and Maria Belk. For contributing to my growth as a mathematician, and for fielding various questions, I would like to thank Sam Hsiao, John Cullinan, Lauren Rose, and Patrick Bahls. Finally, to my family, Jack, Diana, and Noah, and my friends, Thomas, Rebecca, Maia, Laura, Chad, Patrick, Paul, Ezra, Wendy, Nadja, Grace, Kai, Zoë, and Zach K., thank you for the support, companionship and inspiration.

1
Introduction

For the physical theory of String theory, which is currently the most promising approach to a unification of gravity and quantum theory, supersymmetry is a prerequisite. Without supersymmetry it seems unlikely that string theory could ever describe the universe we live in; the only known version of string theory that has the potential to reproduce particles of the Standard Model incorporates supersymmetry. Neither String theory nor supersymmetric theories have been experimentally proven, and neither is wholly understood. For this reason, the theories remain somewhat controversial. However, supersymmetry accounts for the unexplainably low mass of the Higgs particle, eliminates a faster than light particle called the tachyon, and may allow a more precise unification of forces than the Standard Model allows. Further, the theories give rise to beautiful mathematics that continues to fascinate both theoretical physicists and mathematicians, and contains many open questions.

In a supersymmetric theory, every particle that makes up matter (a fermion) corresponds to a partner particle, or superpartner, that transmits a force (a boson) with the same mass and charge. A supersymmetry transformation turns a fermion, a particle with half integer spin, into its superpartner, a boson with integer spin. In the 1970s, Ramond *et al* showed that allowing supersymmetry transformations on particles leaves the laws of physics unchanged, hence it is a symmetrical theory. Physicists now believe that supersymmetry is not exact: supersymmetry must be spontaneously broken; if there were superpartners with the same mass as known particles, we would have observed them. If supersymmetry is broken, superpartners would have greater masses, allowing us to observe them only at high energies. Many physicists suspect that the experiments in the Large Hadron Collider in Geneva are our most likely means of getting real evidence that supersymmetry exists. The reader interested in supersymmetry should refer to [9].

In Mathematics, the relationship between bosons and fermions is encoded in a supersymmetry algebra. Bosonic fields commute, whereas fermionic fields anticommute, and the two fields are incorporated into a single algebra by introducing a $\mathbb{Z}_2$ grading, by which bosons correspond to even elements, and fermions to odd elements. An algebra including a $\mathbb{Z}_2$ grading is called a superalgebra.

There are two approaches to supersymmetrical representations: "on shell," and "off shell." On shell representations are constructed with fields constrained to satisfy differential equations corresponding to laws of motion; these constraints are not given in an off shell representation. Unlike off shell representations, which still pose a challenge to mathematicians, on shell supersymmetry is well understood. However, on shell representations are incredibly difficult to quantize whereas quantizing off shell representations is very straightforward.

In their 2004 paper, [5], Faux and Gates define Adinkra graphs: graphs that visually represent off-shell N-extended one-dimensional supersymmetry; N-extended supersymmetries are extensions of the $(1|1)$ supermultiplets, which are combined to form more intricate structures with additional supersymmetries. Fortunately, the representation of supersymmetry in multiple dimensions is encoded in the representation theory of one-dimensional supersymmetry algebras, ergo we restrict our investigations to supersymmetry algebras in one dimension [6].

The name Adinkra is indigenous to Western Africa; Adinkra symbols have traditionally been used to visually encapsulate concepts, proverbs or philosophical ideas. Likewise, Adinkra graphs allow a visual representation, and a graph theoretical interpretation, that may permit a deeper understanding of the supersymmetrical structure they represent.

In this paper, our main concern is the classification of Adinkra graphs. In Chapter 2, we define Adinkra graphs and consider the question of what determines a unique Adinkra. We cite a theorem by which all Adinkras of the same topology are related, and by which any Adinkra graph having N edge colors can be obtained from the N cube graph, later referred to as the Free Adinkra. In Chapter 3, we introduce homology, an algebraic invariant, as a tool for classifying Adinkras. We find that the dimension of the homology on Adinkra graphs is an infinite sequence given by a polynomial. For any Adinkra, A, there is a polynomial, $p(x)$, such that $p(n)$ gives the n^{th} term in the sequence of Betti numbers for A. In Chapter 4, we develop a deeper understanding of the polynomials corresponding to homology on Adinkras, generating a spanning set for these polynomials, and finally, provide a single equation giving the polynomial corresponding to any Adinkra graph.

2
Adinkra Graphs

2.1 Adinkra Graphs

Definition 2.1.1. An **Adinkra** is a finite, directed graph with the following properties:

1. There is a bipartition of the set of vertices; each vertex is either black or white, and every edge is incident to a black and a white vertex.

2. The graph is edge-N-partite with N distinct edge colors, i.e. each vertex is incident to N distinct edges.

3. There exists a height assignment, a function hgt: $V \to \mathbb{Z}$ such that for any edge going from vertex a to vertex b, $hgt(b) = hgt(a) + 1$.

4. There is an edge parity assignment $\pi : E \to \{-1, 1\}$. (We denote the negative edges with dashed lines.)

5. Finally, every pair of edge colors $\{a, b\}$ incident to a single vertex is part of a 4-cycle of alternating edge colors, such that the edge colors alternate *abab*, and exactly three of the four edges have the same parity.

△

Since a height assignment is given for an Adinkra, in our graphs we will arrange the vertices in rows corresponding to height, such that all vertices of height $n \in \mathbb{N} \cup \{0\}$ are in the same row, and vertices with height $n+1$ are in the row directly below. Therefore, since our graphs are bipartite, the vertices in a given row are either all black or all white, and row color alternates from top to bottom. By arranging the vertices in rows by height, we no longer need directed edges; each edge is directed down.

Definition 2.1.2. A vertex of maximal height, i.e. a vertex which every edge incident is directed towards, is called a **sink**. A vertex of minimal height, i.e. a vertex which every edge incident is directed against, is called a **source**. △

Example 2.1.3. The Adinkra in Figure 2.1.1 has 3 edge colors, i.e. $N = 3$. Note that the graph is bipartite: each edge is incident to a black vertex and a white vertex, and consequently all vertices sharing a color are in the same row, satisfying Condition 1. Also, each vertex is incident to exactly one red, green, and orange edge, satisfying Condition 2.

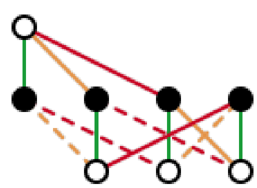


Figure 2.1.1. An Adinkra with 3 edge colors

There is one vertex at height 0; there are 4 and 3 vertices at heights 1 and 2 respectively. The vertex at height 0 is a source, as is the rightmost vertex at height 1. All three vertices at height 2 are sinks. And if we follow an edge from a vertex at height n, we obtain a vertex at height $n+1$. Finally, the reader should convince herself that any 4 cycle with two alternating edge colors has either 3 dashed edges, or 1 dashed edge, so that Condition 5 is satisfied. ◊

2.2 Vertex Lowering and the Hanging Gardens Theorem

If we ignore the height assignments of the nodes and the parities of the edges, the isomorphism class of the graph we are left with is the Adinkra's **topology**. For the time being, we will ignore edge parity. We cite the following theorem from [3].

Theorem 2.2.1. ***(The Hanging Gardens Theorem)*** *Suppose we are given a topology of an Adinkra; a bipartioning of the vertices; a subset S of the vertices which consists of at least one vertex from each connected component; and a height function* $h : S \to \mathbb{Z}$ *such that h(v) is odd if the vertex is black, h(v) is even if the vertex is white, and for every* $s_1, s_2 \in S$ *such that* $s_1 \neq s_2$*,* $dist(s_1, s_2) > |h(s_1) - h(s_2)|$*, where* $dist(s_1, s_2)$ *corresponds to the length of the minimal path from* s_1 *to* s_2*. Up to parity, the Adinkra with this topology, the set of sources S, and their respective height values, is unique.*

Note that if we are given the same information, we can determine a unique Adinkra with the given topology, set of *sinks S*, and *their* respective height values.

Example 2.2.2. The Adinkra in Example 2.1.3, Figure 2.1.1, has two sources; one source is at height 0, the other at height 1. Note that the minimal path between these two sources has length 3, whereas the difference between heights is only 1, hence $\text{dist}(s_1, s_2) > |h(s_1) - h(s_2)|$. $\diamondsuit$

Another theorem that will be of even greater use involves operations called "vertex raising" and "vertex lowering".

Definition 2.2.3. Suppose we are given an Adinkra, and v, a sink or a source in that Adinkra. The operation of reversing the orientation of every arrow incident with v is called **vertex lowering** if v is a source, or **vertex raising** if v is a sink. $\triangle$

Performing the vertex lowering or vertex raising operation makes a sink into a source, or a source into a sink. By lowering a source at height n, the vertex becomes a sink at height $n+2$.

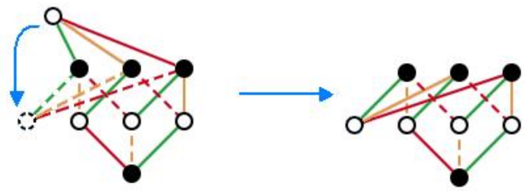


Figure 2.2.1. Example of vertex lowering

Likewise, raising a sink at height n results in a source at height $n-2$. Attempting to raise or lower a vertex that is not a source or a sink violates Condition 3.

In the example in Figure 4.8.1, the vertex lowering operation leaves us with 3 new sinks; our original Adinkra had only 1 sink. Raising or lowering a vertex guarantees a distinct Adinkra graph.

By performing a vertex lowering or vertex raising operation on an Adinkra, we obtain a new Adinkra with the same topology. Since, by the Hanging Gardens Theorem, a set of sources and the heights assigned to those sources determines a unique Adinkra, an Adinkra with a different set of sources must be distinct. The following theorem from [3] ensures that any Adinkra with a given topology can be obtained by performing these operations on another Adinkra with the same topology.

Theorem 2.2.4. *Any two Adinkras with the same topology can be related through a finite sequence of vertex raisings or lowerings.*

Theorem 2.2.4 gives an easy way to determine whether two graphs have the same topology. However for two Adinkras to be equivalent, more requirements must be met. Given two Adinkras, we can ask whether the Adinkras are isomorphic and whether they are in the same equivalence class. For two Adinkras to be *isomorphic*, there must be an isomorphism from one graph to the other such that edge color, edge parity, and height are preserved, and the underlying graphs are isomorphic. For equivalence, we relax the condition that the edge parity be preserved. For two Adinkras to be in the same *equivalence class*, we must be able to transform one Adinkra into the other through a finite number of even shifts in heights, and/or operations reversing the edge parity of all edges incident to a single vertex. Figure 2.2.2 makes the distinction more clear. The two graphs are clearly in the same equivalence class because all we have done is reversed the edge parity of a single vertex. However, because the edge parity is not preserved, the graphs are not isomorphic.

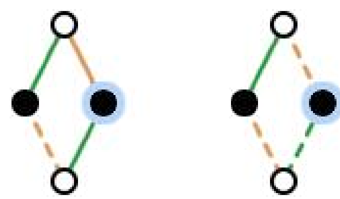


Figure 2.2.2. The Adinkra on the right is obtained from the left by reversing the edge parity of the edges adjacent to the selected vertex.

For two Adinkras to be isomorphic they must be in the same equivalence class, and preserve edge parity. The Adinkras in Figure 2.2.3 are clearly isomorphic because we can obtain one from the other by switching the positions of the two leftmost black vertices; as defined, the order of the vertices in each row of an Adinkra is not significant. Given an Adinkra, we can obtain isomorphic graphs by permuting the vertices in a horizontal row and permuting the edge colors.

Figure 2.2.3. These Adinkras are isomorphic.

The two Adinkras in Figure 2.2.1 are distinct for very clear reasons: the Adinkra on the left has one source and one sink, whereas the Adinkra on the right has three sinks and two sources.

In some cases, however, determining whether two Adinkras are distinct can be significantly more difficult.

For example, in Figure 2.2.4, we have two Adinkras with the same topology, and the same number of sources at the same respective heights. The Adinkras are equivalent, however it is very difficult to determine. To do so we must find a permutation of vertices in each row, a permutation of edge colors, and a set of vertices on which to perform a parity-reversing operation.

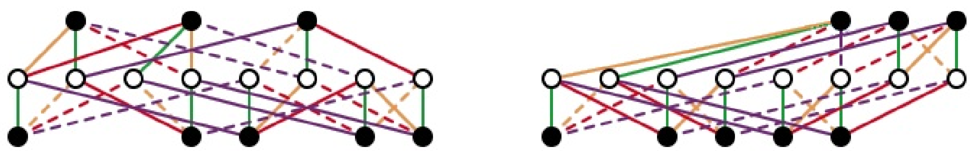


Figure 2.2.4. Equivalent Adinkras.

In [7] and [11], it was suggested that Adinkras be classified by the number of vertices at each height. While this classification would be sufficient to determine equivalence for Adinkras with $N \leq 3$ edge colors, in general, it is not adequate. To show that this classification is not comprehensive, a counterexample is provided in [4]: two Adinkras with 5 edge colors, each of which has 2 vertices at height 0, 8 vertices at height 1, and 6 vertices at height 2; and only one of which has sources at height 1. We provide a simpler counterexample: the Adinkras with 4 edge colors in Figures 2.2.5 and 2.2.6.

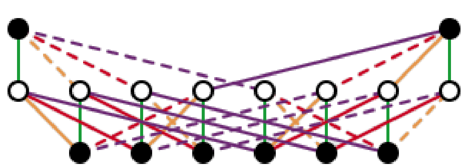


Figure 2.2.5.

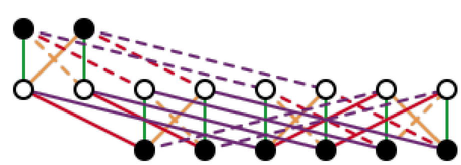


Figure 2.2.6.

Example 2.2.5. The Adinkras in Figures 2.2.5 and 2.2.6 have the same number of vertices in each row and the same number of sinks in each row. Therefore, the vertical reflections of each of these Adinkras are Adinkras whose number of vertices in each row, and number of sources in each row agree. However the Adinkra in Figure 2.2.6 has two sources at height 1 whereas the Adinkra in Figure 2.2.5 has no sources at height 1, so by the Hanging Gardens Theorem, the Adinkras are not equivalent. ♢

Throughout this paper, we will consider the equivalence of Adinkras. We ignore edge parity in the graphs − for our purposes, the information is not relevant.

To date, there is no efficient means by which to classify Adinkra graphs. Currently, the Hanging Gardens Theorem may be the best tool we have for determining equivalence. By the Hanging Gardens Theorem, given two Adinkras with differing numbers of vertices at any height, or differing number of sources at any height, the Adinkras are not equivalent. However, if the preceding information agrees for both Adinkras, determining equivalence is significantly more difficult. The least arduous method for determining equivalence is comparing the sources of the two Adinkras. By the Hanging Garden's Theorem, the set of sources, and their heights determine the graph. If we can find an isomorphism of the graph that maps the sources of one Adinkra to the sources of the other, then the Adinkras are equivalent. However, doing so can be very difficult for large graphs; the hope is that we can find an effective way to classify Adinkra graphs.

3
Homology on Adinkra Graphs

3.1 Homology

In many fields, homology refers to a process by which a group of similar objects or ideas are compared by making evident the places in which they are the same. In mathematics, homology is an algebraic invariant, which can be computed in different ways for different objects. In the following section, we define an exact sequence; the homology of a sequence gives a measure of the degree to which a sequence fails to be exact. Generally, homology can be a useful device for obtaining information about a mathematical object.

We begin with several definitions that motivate an understanding of homology, and discuss two distinct methods by which the homology of an object can be constructed. Later, we examine the homology of Adinkra graphs specifically, first by outlining an approach specific to homology on Adinkras, and then by examining properties that come out of this approach.

The point of interest upon constructing a homology is its dimension at each stage in the sequence. This is given by the Betti number. The Betti numbers of Adinkra graphs have a particularly nice structure; in Section 3.8 we show that the sequence of Betti numbers corresponding to an Adinkra is always given by a polynomial.

3.2 Modules, Exact Sequences and Free Resolutions

The following definitions from [10] motivate our discussion in Sections 3.3 and 3.5 of computing homology.

Definition 3.2.1. Let R be a commutative ring with identity. We call the elements of R scalars and define an R**-module** (or module over R) to be a nonempty set M with two operations: addition and scalar multiplication. The additive operation maps elements $(u,v) \in M \times M$ to $u+v \in M$, i.e. it is a map $M \times M \to M$. Scalar multiplication maps elements $(r,v) \in R \times M$ to $rv \in M$; it is a map $R \times M \to M$. We say that M is a module providing the following conditions hold:

1. M is an abelian group under addition.
2. For all $r,s \in R$, and $u,v \in M$, we have:
   (a) Distributivity of scalar multiplication with respect to addition in M: $r(u+v) = ru+rv$,
   (b) Distributivity of scalar multiplication with respect to addition in R: $(r+s)u = ru+su$,
   (c) Compatibility of scalar multiplication with multiplication in R: $(rs)u = r(su)$,
   (d) Identity element of scalar multiplication: $1u = u$.

△

A module over a field is a vector space.

Definition 3.2.2. A **chain complex**, $(A_\bullet, d_\bullet)$ is a sequence of modules:

$$\cdots \to A_{n+1} \to A_n \to \ \cdots \ \to A_2 \to A_1 \to A_0 \to A_{-1} \to A_{-2} \to \cdots$$

connected by homomorphisms, $d_n : A_n \to A_{n-1}$ such that the composition of any two consecutive maps is zero; i.e. $d_n \circ d_{n+1} = 0$ for all $n \in \mathbb{Z}$. Such a map d_n is called a **differential**. △

Definition 3.2.3. The n^{th} **homology of** $(A_\bullet, d_\bullet)$ is the quotient module $H_n(R) = ker(d_n)/im(d_{n+1})$. The **Betti number**, b_n, is the dimension of the homology, i.e. $b_n = dim(H_n)$. △

A Betti number is usually given by a single number, although in some cases it is given by a list of numbers. For Adinkras with sources at non-zero heights, b_0 is given by a list. We will see cases of this in this chapter and in Chapter 4. However, for all Adinkras, $b_n \in \mathbb{N} \cup \{0\}$ for $n > 0$.

Definition 3.2.4. Let $(A_\bullet, d_\bullet)$ be a chain complex. Let $n \in \mathbb{Z}$. The sequence is an **exact sequence** if $im(d_{n+1}) = ker(d_n)\ \forall\ n \in \mathbb{Z}$. △

For an exact sequence, $H_n = 0$ for all n. So for a given chain complex, the homology gives a measure of the sequence's failure to be exact.

Our work will involve exact sequences of free modules.

Definition 3.2.5. Let M be an R-module. An R-module is a **free module** if $M = \{0\}$, or if M has a basis, i.e. there is a linear independent subset of M that spans M, where the definitions of basis, linear independence, and span are the same as for a vector space. △

Example 3.2.6. Let R be a ring. We denote the Cartesian power of R by R^n, where

$$R^n = \underbrace{R \times R \times R \cdots \times R}_{n} = \{(r_0,\ \ldots,\ r_{n-1}) \mid r_i \in R \text{ for } i \in \mathbb{Z}_n\}.$$

We define multiplication and addition in R^n as follows:

Let $r \in R$, let $(r_1, \ldots, r_n), (s_1, \ldots, s_n) \in R^n$. Then $r(r_1, \ldots, r_n) = (rr_1,\ \ldots,\ rr_n)$, and $(r_1, \ldots, r_n) + (s_1, \ldots, s_n) = (r_1 + s_1,\ \ldots,\ r_n + s_n)$.

With the preceding definition of multiplication and addition on R^n, it is fairly straightforward to deduce that R^n is a module.

Since there is a basis for R^n, namely

$$\{(1,0,0,\ldots,0),\ (0,1,0,\ldots,0),\ (0,0,1,\ldots,0),\ \ldots,\ (0,0,\ldots,0,1)\},$$

R^n is a free module. ◊

Definition 3.2.7. An exact sequence of modules over a ring,

$$\cdots \to C_2 \to C_1 \to C_0 \to M \to 0,$$

with all C_i 's free modules is a **free resolution** of M. △

Finally, we define the tensor map based on [10, Chapter 14].

Definition 3.2.8. Let U,V be vector spaces. Let $\{e_i \mid i \in I\}$ be a basis for U, and let $\{f_j \mid j \in J\}$ be a basis for V. For each ordered pair (e_i, f_j), we invent a "new formal symbol", written $e_i \otimes f_j$, and define T to be the vector space with basis $\{e_i \otimes f_j \mid i \in I, j \in J\}$. The **tensor map**, $t : U \times V \to U \otimes V$, is defined by $t(u,v) = \sum \alpha_i \beta_j (e_i \otimes f_j)$, for $u \in U, v \in V$, with $u = \sum \alpha_i e_i$, and $v = \sum \beta_j f_j$. △

Having covered the preceding definitions, we are now well equipped for our discussion of homology.

3.3 Computing Homology

We provide the general method for computing the homology of a module. The interested reader can refer to the introduction of [2] for a more thorough discussion.

For modules A and B over a ring R, the homology, $H_i = Tor_i^R(A,B)$ is given by the following construction:

1. Take a free resolution of A by R-modules:

$$\cdots \to R^{a_2} \to R^{a_1} \to R^{a_0} \to A \to 0,$$

   $a_i \in \mathbb{N}$.

2. In the next step, replace R with B:

$$\cdots \to B^{a_2} \to B^{a_1} \to B^{a_0}.$$

3. Finally, we take the homology of this chain complex, i.e. ker/im.

Theorem 3.3.1. *The homology of modules is commutative, i.e.,*

$$Tor_i^R(A,B) \cong Tor_i^R(B,A).$$

The above theorem is found in [2].

In the following section we provide definitions of the exterior algebra, Λ^* and of the symmetric algebra, Sym^*, which are necessary for our discussion in Section 3.5 of homology on Adinkras.

3.4 The Exterior Algebra and Symmetric Algebra

For an Adinkra, to compute the homology and its corresponding dimensions, we construct a free resolution of M over R where R is the exterior algebra, and M is an R-module. We provide the following definitions based on [10].

Definition 3.4.1. Let V be a finite dimensional vector space, with basis $\{x_1, \ldots,\ x_n\}$. For $k \in \mathbb{N}$, let $T^k(V)$ be the free tensor algebra with basis

$$\{x_{i_1} \otimes \cdots \otimes x_{i_k}\}_{i_1, i_2, \ldots, i_k \in \{1,2,\ldots,n\}},$$

and let

$$T^*(V) = T^0(V) \oplus T^1(V) \oplus T^2(V) \oplus \ldots = \bigoplus_{p \in \mathbb{Z} \geq 0} T^p(V).$$

△

Definition 3.4.2. We define the **wedge product**, denoted $\wedge$, by $a \wedge b = \frac{1}{2}(a \otimes b - b \otimes a)$. △

Note that $a \wedge b = \frac{1}{2}(a \otimes b - b \otimes a) = -\frac{1}{2}(b \otimes a - a \otimes b) = -b \wedge a$.

Definition 3.4.3. We let V be a finite dimensional vector space over $\mathbb{R}$. The **exterior algebra**, is given by the quotient $\Lambda^*(V) = T^*(V)/\langle a \otimes b + b \otimes a \rangle$, for $a, b \in V$. We denote the exterior algebra $\Lambda^* = \Lambda^0 \oplus \Lambda^1 \oplus \Lambda^2 \oplus \ldots \oplus \Lambda^n$, with Λ^k = span of all k-fold products of $\{e_1, \ldots, e_n\}$. The basis for the exterior algebra is given by

$$\{x_{i_1} \wedge \cdots \wedge x_{i_k}\}_{i_1 < i_2 < \ldots < i_k \in \{1,2,\ldots,n\}}.$$

for $k \leq n$. △

By definition, the exterior algebra is a quotient ring, hence it is a ring.

Definition 3.4.4. The **free Adinkra** is a graph that gives a description of the exterior algebra of $span\{e_1,\dots,e_N\}$. The vertices of the free Adinkra correspond to elements of the basis of the exterior algebra, and the edges of the free Adinkra correspond to the wedge operation with $e_1, e_2, \dots, e_N$. Since the exterior algebra is a ring, the free Adinkra corresponds to a ring. △

Following an edge e_j down from a vertex e_i gives the vertex $e_j \wedge e_i$.

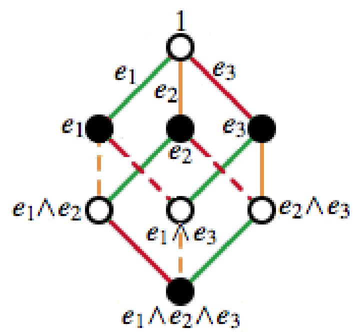


Figure 3.4.1. The figure shows a free Adinkra with 3 edge colors: e_1, e_2, e_3, denoted with green, orange, and red lines respectively. If we follow the edges e_1 then $-e_2$, or e_2 then e_1, we obtain the vertex $e_1 \wedge e_2$. All vertices are given by the preceding construction.

Example 3.4.5. Let $N = 3$, i.e. there are edges of types $\{e_1, e_2, e_3\}$. The exterior algebra is given by

$$\Lambda^* = \Lambda^0 \oplus \Lambda^1 \oplus \Lambda^2 \oplus \Lambda^3.$$

It is always the case that

$$\Lambda^0 = \text{span}\{1\}.$$

We observe

$$\Lambda^1 = \text{span}\{\text{e}_1, \text{e}_2, \text{e}_3\},$$

and compute

$$\Lambda^2 = \text{span}\{\text{e}_1 \wedge \text{e}_2,\ \text{e}_1 \wedge \text{e}_3,\ \text{e}_2 \wedge \text{e}_3\},$$

and

$$\Lambda^3 = \text{span}\{\text{e}_1 \wedge \text{e}_2 \wedge \text{e}_3\}.$$

Therefore the basis for the exterior algebra is

$$\beta = \{1,\ e_1,\ e_2,\ e_3,\ e_1 \wedge e_2,\ e_1 \wedge e_3,\ e_2 \wedge e_3,\ e_1 \wedge e_2 \wedge e_3\},$$

and the free Adinkra is the Adinkra corresponding to the exterior algebra, shown in Figure 3.4.1. Note that since $e_i \wedge e_i = \frac{1}{2}(e_i \otimes e_i - e_i \otimes e_i) = 0$, there are no downward edges e_i from vertices e_i. Likewise, there are no downward edges e_i or e_j from vertices $e_i \wedge e_j$. Consequently the graph is 3-regular. $\diamondsuit$

In general, the free Adinkra with n edge colors is an n-regular graph for which the k^{th} row corresponds to Λ^k, hence the number of vertices in the k^{th} row is given by $\binom{n}{k}$. This property will be of interest to us later, particularly in Section 3.8.

Finally, we define the symmetric algebra, which we use in our construction of homology in the following section.

Definition 3.4.6. Let V be an n dimensional vector space over $\mathbb{R}$. The symmetric algebra, is given by the quotient $Sym^*(V) = T^*(V)/\langle a \otimes b - b \otimes a \rangle$, for $a, b \in V$, where

$$Sym^*(V) = \bigoplus_{p \in \mathbb{Z} \geq 0} Sym^p(V),$$

and the basis for the symmetric algebra is given by

$$\{x_{i_1} x_{i_2} \cdots x_{i_k}\}_{i_1 \leq i_2 \leq \ldots \leq i_k \in \{1,2,\ldots,n\}},$$

for $k \in \mathbb{N}$. $\triangle$

3.5 Homology on Adinkras

For Adinkras, we compute the homology given modules M and $\mathbb{R}$, where M is a module over Λ^*. First we compute $H_i = Tor_i^{\Lambda^*}(\mathbb{R}, M)$, which is given by the following construction:

1. Take the following free resolution of $\mathbb{R}$ by Λ^*-modules:

$$\cdots \to \Lambda^* \otimes Sym^3\{e_i\} \to \Lambda^* \otimes Sym^2\{e_i\} \to \Lambda^* \otimes \{e_i\} \to \Lambda^* \to \mathbb{R} \to 0.$$

This free resolution is a standard one for Lie algebra homology [2].

2. Next, replace Λ^* with M:

$$\cdots \to M \otimes Sym^3\{e_i\} \to M \otimes Sym^2\{e_i\} \to M \otimes \{e_i\} \to M.$$

3. Now, take the homology, i.e. ker/im.

It turns out that, for our purposes, the construction of $Tor_i^{\Lambda^*}(M,\mathbb{R})$ is easier to compute, though generally, this is not the case. The construction of $H_i = Tor_i^{\Lambda^*}(M,\mathbb{R})$ is as follows:

1. Take a free resolution of M by Λ^*-modules:

$$\cdots \to \Lambda^{*a_2} \to \Lambda^{*a_1} \to \Lambda^{*a_0} \to M \to 0.$$

Note that the free resolution is always an exact sequence.

2. Replace Λ^* with $\mathbb{R}$:

$$\cdots \to \mathbb{R}^{a_2} \to \mathbb{R}^{a_1} \to \mathbb{R}^{a_0}.$$

3. If the free resolution is minimal, then all maps are the zero map, and $a_i = b_i$, where b_i is the Betti number. Then, we take the homology and obtain:

$$\ldots, \mathbb{R}^{b_2}, \mathbb{R}^{b_1}, \mathbb{R}^{b_0},$$

where $\mathbb{R}^{b_i} = H_i$.

See [2] for a more detailed explanation.

It is often very difficult to compute minimal free resolutions; however, in Section 3.7 we describe a method that does so automatically.

3.6 Multiplication on Adinkras

In this section we define multiplication on Adinkras, both algebraically and graphically. From Sections 3.2 and 3.4, we know that a free Adinkra R corresponds to a ring. For any Adinkra A, we define the product of R and A, $R \times A \to A$ giving the module structure of A over R. If $A = R$, we have $R \times R \to R$ which gives the product structure of the ring. Then every Adinkra A corresponds to a module over the exterior algebra, Λ^*.

Definition 3.6.1. Let $A,\ R$ be Adinkras, where R is the free Adinkra. Taking the **product** $v_j(v_i)$ for $v_i \in A,\ v_j \in R$ is equivalent to following the same sequence of edge colors from v_i to $v_j(v_i)$ that we follow from 1 to v_j. △

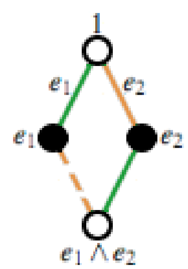


Figure 3.6.1. The free Adinkra, R

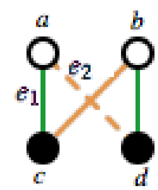


Figure 3.6.2. The Adinkra, A

$\times$	a	b	c	d
1	a	b	c	d
e_1	c	d	0	0
e_2	$-d$	c	0	0
$e_1 \wedge e_2$	0	0	0	0

Table 3.6.1. $R \times A$

$\times$	1	e_1	e_2	$e_1 \wedge e_2$
1	1	e_1	e_2	$e_1 \wedge e_2$
e_1	e_1	0	$e_1 \wedge e_2$	0
e_2	e_2	$e_1 \wedge e_2$	0	0
$e_1 \wedge e_2$	$e_1 \wedge e_2$	0	0	0

Table 3.6.2. $R \times R$

Example 3.6.2. Let R be the free Adinkra with two edge colors, given in Figure 3.6.1, let M be the Adinkra in Figure 3.6.2. For all vertices $v_i \in R,\ v_j \in A$, we compute $v_j(v_i)$ in Table 3.6.1. ◊

Example 3.6.3. We denote the Adinkra in Figure 3.6.1 by R. For all vertices $v_i,\ v_j$ in R, we compute $v_j(v_i)$ in Table 3.6.2. ◊

As in Examples 3.6.2 and 3.6.3, we can generally compute $R \times A$ and $R \times R$ for a free Adinkra R. Note that $A \times A$ is not necessarily well defined. For example, in the Adinkra in Figure 3.6.2 there are two edges that we can follow to get the vertex c, and likewise for the vertex d.

3.7 Computing the Homology of an Adinkra

In Section 3.5, we pointed out that there are two constructions by which we can compute the homology of an Adinkra. To compute the homology of an Adinkra using the first construction, we take the module M, given by the Adinkra, and define the function

$$d_k : M \otimes Sym^k\{e_i\} \to M \otimes Sym^{k-1}\{e_i\}$$

by

$$d_k(v \otimes e_{i_1} \cdots e_{i_k}) = e_{i_1}(v) \otimes e_{i_2} \cdots e_{i_k} + \ldots + e_{i_k}(v) \otimes e_{i_1} \cdots e_{i_{k-1}}.$$

Example 3.7.1. We compute the homology of the Adinkra, M, in Figure 3.6.2 using the preceding construction. In this example, our function $d_1(v \otimes e_i) = e_i(v)$.

$$\begin{array}{llll} d_1(a\otimes e_1) = c & d_1(b\otimes e_1) = d & d_1(c\otimes e_1) = 0 & d_1(d\otimes e_1) = 0 \\ d_1(a\otimes e_2) = d & d_1(b\otimes e_2) = -c & d_1(c\otimes e_2) = 0 & d_1(d\otimes e_2) = 0 \end{array}$$

For $d_2(v \otimes e_i e_j) = e_i(v) \otimes e_j + e_j(v) \otimes e_i$, we compute

$d_2(a \otimes e_1 e_2) = (c \otimes e_2) + (d \otimes e_1)$,

$d_2(b \otimes e_1 e_2) = (d \otimes e_2) + (-c \otimes e_1)$,

$d_2(c \otimes e_1 e_2) = 0$,

$d_2(d \otimes e_1 e_2) = 0$.

We take the homology, $H_1 = ker(d_1)/im(d_2)$. The kernel of d_1 is $span\{c \otimes e_1, d \otimes e_1, c \otimes e_2, d \otimes e_2\}$, and the image of d_2 is $span\{(c \otimes e_2 + d \otimes e_1), (d \otimes e_2 - c \otimes e_1)\}$. We compute $b_1 = dim(H_i)$:

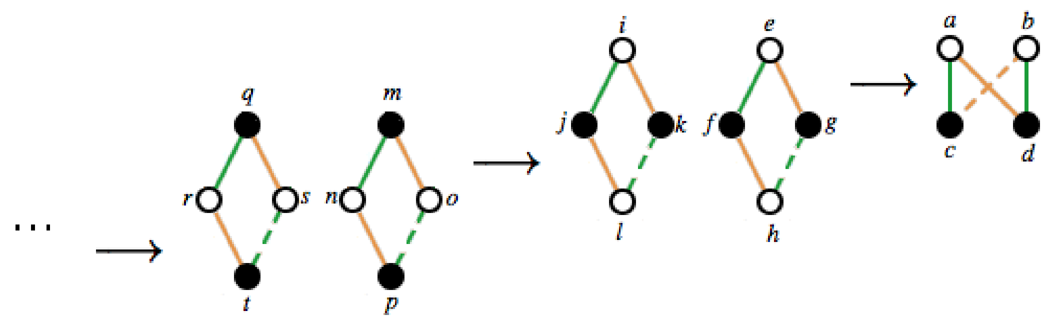
$$dim(H_1) = dim(ker(d_1)) - dim(im(d_2)) = 4 - 2 = 2.$$

When we take the kernel modulo the image, we have $c \otimes e_2 \sim -d \otimes e_1$, and $d \otimes e_2 \sim -c \otimes e_1$, therefore the dimension of H_1 is 2. ◊

Next, we compute the homology through the alternate construction.

Example 3.7.2. We compute the homology of the Adinkra, M, in Figure 3.6.2 using the following construction: We take the free resolution of modules Λ^{*a_i} for $a_i \in \mathbb{N}$, where Λ^* is the exterior algebra corresponding to the free Adinkra with the same number of edge colors as our Adinkra. Our function d is a homomorphism. It is surjective to preserve exactness.

Below we have the free resolution $\cdots \to \Lambda^{*2} \to \Lambda^{*2} \to M$. Our function d is as follows:



$d(i) = a$	$d(j) = -c$	$d(q) = j+g$	$d(r) = -h$
$d(e) = b$	$d(k) = d$	$d(m) = k-f$	$d(s) = l$
	$d(f) = d$		$d(n) = -l$
	$d(g) = c$		$d(o) = h$

By replacing Λ^* with $\mathbb{R}$ we eliminate all vertices but the top vertex. Since the top vertices of the free Adinkras are at different heights, $\ldots \to \mathbb{R}^2 \to \mathbb{R}^2$ corresponds to the zero map, hence the free resolution is minimal. Therefore the Betti numbers are simply the number of free Adinkras at each stage in the sequence, i.e. $\ldots, 2, 2$. ◊

Using the second construction, the Betti numbers, b_i, for $i \in \mathbb{N} \cup 0$, can be calculated as follows: Let M be an Adinkra with n edge colors and height k, with a_0 sources, such that a_0 vertices have height 0, a_1 vertices have height $1, \ldots, a_k$ vertices have height k. If M is the free Adinkra then

$b_0 = 1$ and $0 = b_1 = b_2 = b_3 = \dots$; otherwise $k < n$ and $b_i = |c_i|$, where c_i is given by

$$\begin{aligned}
c_0 &= a_0,\\
-c_1 &= c_0\binom{n}{1} - a_1,\\
-c_2 &= c_0\binom{n}{2} + c_1\binom{n}{1} - a_2,\\
-c_k &= c_0\binom{n}{k} + c_1\binom{n}{k-1} + \dots + c_{k-1}\binom{n}{1} - a_k,\\
-c_{k+1} &= c_0\binom{n}{k+1} + c_1\binom{n}{k} + \dots + c_k\binom{n}{1},\\
&\dots,\\
-c_n &= c_0\binom{n}{n} + c_1\binom{n}{n-1} + \dots + c_{n-1}\binom{n}{1}.
\end{aligned}$$

Likewise for $j \in \mathbb{N}$ such that $j \geq 1$,

$$-c_{n+j} = c_j\binom{n}{n} + c_{j+1}\binom{n}{n-1} + \dots + c_{j+n-1}\binom{n}{1}.$$

Hence, for $j \geq 0$,

$$c_j\binom{n}{n} + c_{j+1}\binom{n}{n-1} + \dots + c_{j+n-1}\binom{n}{1} + c_{j+n}\binom{n}{0} = 0. \tag{3.7.1}$$

That is,

$$(c_j, c_{j+1}, \dots, c_{j+n}) \cdot \left(\binom{n}{n}, \binom{n}{n-1}, \dots, \binom{n}{0}\right) = (c_j, c_{j+1}, \dots, c_{j+n}) \cdot \left(\binom{n}{0}, \binom{n}{1}, \dots, \binom{n}{n}\right) = 0.$$

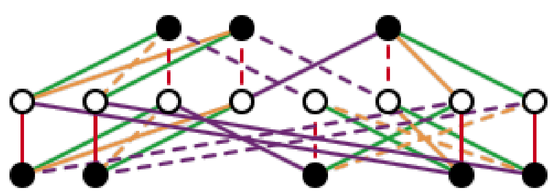


Figure 3.7.1.

Example 3.7.3. Let M be the Adinkra in Figure 3.7.1, with 4 edge colors. Note there are 3 rows of 3, 8, and 5 vertices respectively from top to bottom. Recall, in Chapter 2, for an edge going from vertex a to vertex b we defined $hgt(b) = hgt(a) + 1$. We assign height 0 to the top row; since our edges are directed down, the second row has height 1, and the bottom row has height 2, hence $a_0 = 3$, $a_1 = 8$, and $a_2 = 5$.

$$\begin{matrix}3\\8\\5\end{matrix} \longleftarrow 3\begin{pmatrix}1\\4\\6\\4\\1\end{pmatrix} \longleftarrow -4\begin{pmatrix}1\\4\\6\\4\\1\end{pmatrix} \longleftarrow 3\begin{pmatrix}1\\4\\6\\4\\1\end{pmatrix} \longleftarrow -5\begin{pmatrix}1\\4\\6\\4\\1\end{pmatrix}$$

We calculate b_i by finding the minimal number of copies of Λ^* for our map to be surjective. Hence b_0 is given by the number of sources. For $i > 0$, we choose b_i so that our sequence preserves exactness. Since the arrows are at different heights, we can see that the resolution is minimal.

Notice that:

$$\begin{pmatrix}3\\8\\5\end{pmatrix} - 3\begin{pmatrix}1\\4\\6\\4\\1\end{pmatrix} = \begin{pmatrix}0\\-4\\-13\\-12\\-3\end{pmatrix} + 4\begin{pmatrix}1\\4\\6\\4\\1\end{pmatrix} = \begin{pmatrix}0\\3\\12\\13\\4\end{pmatrix} - 3\begin{pmatrix}1\\4\\6\\4\\1\end{pmatrix} = \begin{pmatrix}0\\0\\-5\\1\\3\end{pmatrix} + 5\begin{pmatrix}1\\4\\6\\4\\1\end{pmatrix} = \begin{pmatrix}0\\21\\33\\20\\5\end{pmatrix} \cdots$$

so our sequence is exact, and the Betti numbers are $3,4,3,5,\ldots$. ◊

3.8 A Polynomial Expression for the Betti Numbers

In this section, we develop the main result of this paper: that the Betti numbers corresponding to Adinkra graphs are given by polynomials.

We observe that since $\binom{n}{k} = \frac{k+1}{n+1}\binom{n+1}{k+1}$,

$$\left(\binom{n}{0},\ \binom{n}{1},\ \ldots,\ \binom{n}{n}\right) = \frac{1}{n+1}\left(\binom{n+1}{1},\ 2\binom{n+1}{2},\ \ldots,\ (n+1)\binom{n+1}{n+1}\right). \tag{3.8.1}$$

This fact will be of use in the following theorem, in which we show that the dot product of a polynomial with degree $\leq n$ evaluated at $n+1$ consecutive integer values and a vector given by ordered elements of the n^{th} row of Pascal's triangle alternating in sign is zero for $n \in \mathbb{N}$:

Theorem 3.8.1. *Let $p(x)$ be a polynomial with degree less than n. Then*

$$(p(x), p(x+1), \ldots, p(x+n)) \cdot \left(\binom{n}{0},\ -\binom{n}{1},\ \ldots,\ \pm\binom{n}{n}\right) = 0.$$

Proof. Our proof is by induction. For our base case, let $p(x)$ be a polynomial of degree 0. Then $p(x) = c$, for some $c \in \mathbb{R}$, and $p(x+m) = c$ for $m \in \mathbb{N}$. Hence

$$(p(x), p(x+1), \ldots, p(x+n)) = (c, c, \ldots, c).$$

Therefore

$$\begin{aligned}
&(p(x), p(x+1), \ldots, p(x+n)) \cdot \left(\binom{n}{0},\ -\binom{n}{1},\ \ldots,\ \pm\binom{n}{n}\right)\\
&= c(1,1,\ldots,1) \cdot \left(\binom{n}{0},\ -\binom{n}{1},\ \ldots,\ \pm\binom{n}{n}\right)\\
&= c\left(\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \ldots \pm \binom{n}{n}\right)\\
&= c(1-1)^n = 0.
\end{aligned}$$

Now we assume the statement is true for n and deduce that it must be true for $n+1$. Let $p(x)$ have degree n. Then

$$\begin{aligned}
&(p(x), p(x+1), \ldots, p(x+n+1)) \\
&= \left(\sum_{k=0}^{n} a_k \binom{n}{k} x^{n-k} 0^k \sum_{k=0}^{n} a_k \binom{n}{k} x^{n-k} 1^k,\ \ldots, \sum_{k=0}^{n} a_k \binom{n}{k} x^{n-k} (n+1)^k\right) \\
&= \sum_{k=0}^{n} a_k \binom{n}{k} x^{n-k} (0^k, 1^k, 2^k,\ \ldots, (n+1)^k).
\end{aligned}$$

We can show that

$$\left(0^k, 1^k, 2^k,\ \ldots, (n+1)^k\right) \cdot \left(\binom{n+1}{0}, -\binom{n+1}{1}, \ldots, \pm\binom{n+1}{n+1}\right) = 0;$$

hence

$$(p(x), p(x+1), \ldots, p(x+n+1)) \cdot \left(\binom{n+1}{0}, -\binom{n+1}{1}, \ldots, \pm\binom{n+1}{n+1}\right) = 0.$$

By our assumption, for $k \in \{0, 1,\ \ldots, n-1\}$,

$$(0^k, 1^k, 2^k,\ \ldots, (n+1)^k) \cdot \left(\binom{n+1}{0}, -\binom{n+1}{1}, \ldots, \pm\binom{n+1}{n+1}\right) = 0,$$

hence

$$\sum_{k=0}^{n} a_k \binom{n}{k} x^{n-k} (0^k, 1^k, 2^k,\ \ldots, (n+1)^k) = (0^n, 1^n, 2^n,\ \ldots, (n+1)^n).$$

However, note that

$$\begin{aligned}
&(0^n, 1^n, 2^n,\ \ldots, (n+1)^n) \cdot \left(\binom{n+1}{0}, -\binom{n+1}{1}, \ldots, \pm\binom{n+1}{n+1}\right) \\
&= \frac{1}{n+1}(0^{n-1}, 1^{n-1}, 2^{n-1},\ \ldots, (n+1)^{n-1}) \cdot \left(\binom{n+1}{0}, -\binom{n+1}{1}, \ldots, \pm\binom{n+1}{n+1}\right) \\
&= 0
\end{aligned}$$

by (3.8.1). □

Example 3.8.2. Let $p(x) = x^2 - 1$.

Since $p(x)$ has degree 2, let $n = 3$. Then

$$(p(x), p(x+1), p(x+2), p(x+3)) = (x^2-1, (x+1)^2-1, (x+2)^2-1, (x+3)^2-1),$$

and,

$$\begin{aligned}
&(1,-3,3,-1)\cdot(x^2-1,(x+1)^2-1,(x+2)^2-1,(x+3)^2-1)\\
&=x^2-1-3(x^2+2x+1-1)+3(x^2+4x+4-1)-(x^2+6x+9-1)\\
&=0.
\end{aligned}$$

Likewise, let $n=4$. We have

$$(p(x),p(x+1),p(x+2),p(x+3))=(x^2-1,(x+1)^2-1,(x+2)^2-1,(x+3)^2-1,(x+4)^2-1),$$

and

$$\begin{aligned}
&(1,-4,6,-4,1)\cdot(x^2-1,(x+1)^2-1,(x+2)^2-1,(x+3)^2-1,(x+4)^2-1)\\
&=x^2-1-4(x^2+2x+1-1)+6(x^2+4x+4-1)-4(x^2+6x+9-1)+x^2+8x+16-1\\
&=0.
\end{aligned}$$

Finally, note that $(1,-2,1)\cdot(p(x),p(x+1),p(x+2))\neq 0$. $\Diamond$

Now we prove the converse:

Theorem 3.8.3. *Let $n\in\mathbb{N}$, and let p be a function such that*

$$(p(x),p(x+1),\ldots,p(x+n))\cdot\left(\binom{n}{0},\,-\binom{n}{1},\,\ldots,\,\pm\binom{n}{n}\right)=0$$

for $x\in\mathbb{N}\cup\{0\}$. Then p is a polynomial.

Proof. Let $n\in\mathbb{N}$, and let

$$\mathcal{S}=\left\{f\mid(f(x),f(x+1),\ldots,f(x+n))\cdot\left(\binom{n}{0},\,-\binom{n}{1},\,\ldots,\,\pm\binom{n}{n}\right)=0\right\}.$$

We show that $\mathcal{S}$ is a vector space. Let $f, g \in \mathcal{S}$ and $a,b \in \mathbb{Q}$. By the properties of the dot product,

$$\begin{aligned}
&((af+bg)(x),(af+bg)(x+1),\ldots,(af+bg)(x+n))\cdot\left(\binom{n}{0},\, -\binom{n}{1},\, \ldots,\, \pm\binom{n}{n}\right)\\
&= a\,(f(x),f(x+1),\ldots,f(x+n))\cdot\left(\binom{n}{0},\, -\binom{n}{1},\, \ldots,\, \pm\binom{n}{n}\right)\\
&\quad + b\,(g(x),g(x+1),\ldots,g(x+n))\cdot\left(\binom{n}{0},\, -\binom{n}{1},\, \ldots,\, \pm\binom{n}{n}\right)\\
&= a0+b0=0.
\end{aligned}$$

Further, note that $(0,\ldots,0),(1,\ldots 1) \in \mathcal{S}$. The other properties of a vector space follow trivially.

Now, let $h \in \mathcal{S}$. Then

$$h(x+n) = \binom{n}{0}h(x) - \binom{n}{1}h(x+1) + \ldots \pm \binom{n}{n-1}h(x+n-1)\ \ \forall\, x \in \mathbb{N}\cup\{0\}.$$

Using the formula above, we can determine h_m for $m \geq n$ inductively as a linear combination of the preceding m values. Hence every function $h \in \mathcal{S}$ is completely determined by $(h(0),h(1),\ldots,h(n-1))$.

Since every function in $\mathcal{S}$ is determined by a vector with n components, the vector space over $\mathbb{Q}$ given by elements in $\mathcal{S}$ has dimension n. Therefore it is isomorphic to the vector space over $\mathbb{Q}$ with basis $\{1,x,x^2,\ldots,x^{n-1}\}$ of degree $n-1$ polynomials with rational coefficients. □

By Theorems 3.8.1 and 3.8.3, and (3.7.1) we can deduce the following:

Corollary 3.8.4. *Given an Adinkra with N edge colors, for $n \geq N$, b_n is determined by a polynomial of degree less than N.*

We have dubbed this polynomial the Betti polynomial.

In 2008, Iga [6] showed that the homology of a vertical reflection of an Adinkra is in fact the cohomology of that Adinkra. Hence the cohomology of an Adinkra also corresponds to a polynomial. And the homology of an Adinkra gives information about the number of sources at each height, while the cohomology gives information about the sinks at each height.

3.9 Computing the Betti Polynomial

To calculate the polynomial corresponding to a given Adinkra, we developed code in Sage; see Appendix C.

Example 3.9.1. To calculate the Betti polynomial for the Adinkra in Figure 3.7.1, we enter `BettiPolynomial(4, [3,8,5])`. The output is x^2-4. If we plot this polynomial, we can obtain the Betti numbers b_n for $n \geq 3$.

The Betti numbers are: 3, 4, 3, 5, 12, 21, 32, 45, 60, 77, 96, 117, 140, 165, $\ldots$. ◊

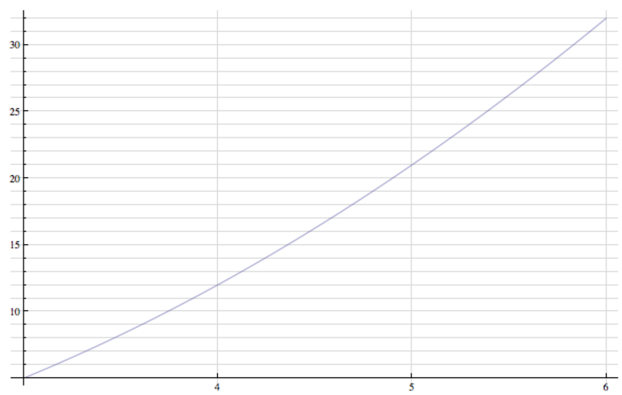


Figure 3.9.1. Graph of $f(x) = x^2 - 4$

Example 3.9.2. Next, we calculate the Betti polynomial for an Adinkra with sources in other rows, such as the Adinkra in Figure 3.9.2. This Adinkra has 3 vertices in the top row, 8 in the second row, and 5 in the bottom row; it has 3 sources in the top row and 1 in the middle row. We input `BettiPolynomial(4, [3,8,5],[3,1])`, and obtain the polynomial $(x+2)(x-2) = x^2-4$.

The Betti numbers are: $(3,1)$, 5, 3, 5, 12, 21, 32, 45, 60, 77, 96, 117, 140, 165, $\ldots$.

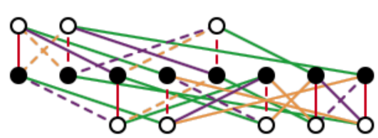


Figure 3.9.2.

Note that b_0 is a list of numbers, $(3,1)$. This is because we compute H_0 by taking 3 copies of the free Adinkra corresponding to the 3 sources at height 0, and one copy of the free Adinkra corresponding to the source at height 1. $\diamondsuit$

4

Understanding Betti Polynomials

4.1 Uniqueness of Betti Polynomials

For an Adinkra graph with N edge colors, we obtain a finite number of distinct Adinkras with the same topology using the vertex lowering or vertex raising operations. In Chapter 3, we showed that each of these Adinkras corresponds to a polynomial $p(x)$ with degree $< N$, with $b_k = p(k)$ for $k \geq N$. In this chapter, we investigate the properties of these polynomials and derive a single equation from which the Betti polynomial for any Adinkra can be obtained.

We introduce the following notation: We let

$$(c_0,\ c_{1_{k_1}}^{s_1},\ c_{2_{k_2}}^{s_2},\ \ldots,\ c_{n-2_{k_{N-2}}}^{s_{N-2}},\ c_{N-1},\ c_N)$$

denote an Adinkra with N edge colors, c_i vertices at height i, s_i sources at height i, and k_i sinks at height i, for $i \in \mathbb{N} \cup \{0\}$. We leave out subscripts and superscripts for $c_0,\ c_{N-1}$, and c_N since for any Adinkra, $s_0 = c_0$, $k_0 = 0$, $k_N = c_N$, and at heights $N-1$ and N there are no sources. Whenever possible, we leave out entries, superscripts and subscripts corresponding to 0.

Example 4.1.1. The Adinkra in Figure 4.1.1 is labeled $(4, 8_1^1, 4)$ since there are 4, 8, and 4 vertices at heights 0, 1, and 2 respectively, and at height 1 there is one source and one sink. $\Diamond$

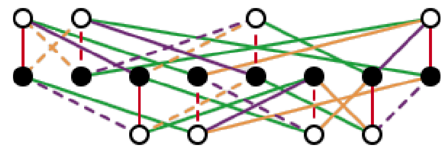


Figure 4.1.1.

In this section, we consider the uniqueness of the polynomials. The Betti polynomials are a function of the number of edge colors, the number of vertices in each row, and the number of sources in each row. However, it is clear that the polynomials are not unique. Namely, Adinkras with different numbers of edge colors may correspond to the same polynomial. The simplest example is the 0 polynomial; the Betti polynomial for a free Adinkra is 0 regardless of the number of edge colors. Likewise, the polynomial $4x+4$ is the Betti polynomial for both the $(4,4)$ Adinkra with 3 edge colors, and the $(4,8,4)$ Adinkra with 4 edge colors. In fact, the set of Betti polynomials for Adinkras with $N=3$ is a subset of the set of Betti polynomials for $N=4$ Adinkras; that is, every Betti polynomial for an $N=3$ Adinkra is also a Betti polynomial for some Adinkra with 4 edge colors. Other such examples can be found in Appendix B.

Fixing N does not guarantee a unique polynomial either. For $N=4$, we can construct Adinkras with 8 or 16 vertices. Similarly, for $N=5$, we can construct Adinkras with 16 or 32 vertices. In fact, it is only the case for $N<4$ that the number of vertices is predetermined; in Appendix A, we provide a table of the number of vertices in Adinkra graphs for $2 \leq N \leq 8$. As a consequence, it is possible to have two distinct Adinkras that agree in number of edge colors, number of vertices at each height, and number of sources at each height. That is, all of the parameters for $b(n)$ are the same, and therefore the polynomials are the same.

Example 4.1.2. There are two distinct Adinkra graphs with 4 edge colors that have 2 vertices at height 0, 8 vertices at height 1, and 6 vertices at height 2, see Figures 4.1.2 and 4.1.3. The first is made up of two $(1,4,3)$ Adinkras, the second is a single connected component with 16 vertices. The Adinkras agree in number of sources, height of sources, number of vertices, and number of edge colors, and the polynomial gives no information on whether the graph is connected, hence the polynomials are the same.

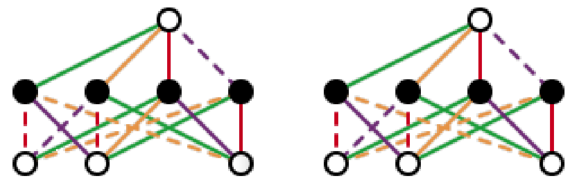


Figure 4.1.2. An Adinkra with 4 edge colors and 16 vertices with Betti polynomial $2x^2 + 4x$

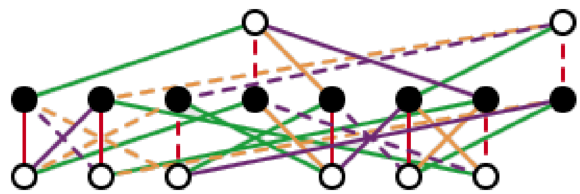


Figure 4.1.3. A connected Adinkra with 4 edge colors and 16 vertices with Betti polynomial $2x^2 + 4x$

Note that the Betti polynomial for a single connected component of the Adinkra in Figure 4.1.2 is $\frac{1}{2}\left(2x^2 + 4x\right) = x^2 + 2x$. The polynomial for an Adinkra with k of these connected components is $k(x^2 + 2x)$. ◊

In addition to being a function of the number of edge colors and the number of vertices in each row, the Betti polynomial is a function of the number of sources. The Betti numbers do not depend on connectedness or on the number of sinks. For example, the $(6,8,2)$ and $(6,8_2,2)$ Adinkras with 4 edge colors have the same polynomial, despite the fact that by the Hanging Gardens Theorem they are not equivalent, since only one of those Adinkras has sinks at height 1. However, their vertical reflections $(2,8,6)$, and $(2,8^2,6)$ have distinct polynomials.

Finally, we note that there are cases in which Adinkras with the same number edge colors, the same number of vertices at each height and sources at differing heights do in fact correspond to the same polynomial.

Example 4.1.3. The $(3,8,5)$ Adinkra and the $(3,8^1,5)$ Adinkra both correspond to the polynomial x^2-4. However, while the Adinkras agree in all other respects, one Adinkra has a source at height 1, while the other does not. ◊

We will return to this example in Section 4.7.

4.2 Polynomials for Valise Adinkras

If we consider the Betti polynomials for "valise" Adinkras, i.e. Adinkras with all vertices in 2 rows, we see an interesting pattern. Since all valise Adinkras are of the form $k(1,1)$ for $k \in \mathbb{N}$, and the Betti numbers corresponding to an Adinkra do not depend on connectedness, we compute the Betti polynomial for (1,1) as if it were an Adinkra with N edges. In Section 1 of this chapter, we saw that to compute the Betti polynomial for an Adinkra with 2 vertices in the top row, 8 in the second, and 6 in the third, we could find the Betti polynomial for the (1,4,3) and multiply it by 2 since 2(1,4,3) = (2,8,6). Likewise, the polynomial for the (k,k) with N edges should be k times the polynomial corresponding to the homology of (1,1) with the free Adinkra with N edges.

Lemma 4.2.1. *Let $N>2$. Then the polynomial for the $(1,1)$ Adinkra is given by*

$$\frac{1}{(N-2)!}(x+1)(x+2)\cdots(x+N-2).$$

Proof. For the $(1,1)$ Adinkra, the Betti numbers are given as follows:

$$
\begin{aligned}
B_0 &= \binom{N}{0} \\
B_1 &= \binom{N}{1} - 1 \\
B_2 &= \left| \binom{N}{2} - B_1 \binom{N}{1} \right| \\
B_3 &= \left| \binom{N}{3} - B_1 \binom{N}{2} + B_2 \binom{N}{1} \right| \\
&\vdots \\
B_N &= \left| \binom{N}{N} - B_1 \binom{N}{N-1} + B_2 \binom{N}{N-2} - B_3 \binom{N}{N-3} + \ldots \pm B_{N-1} \binom{N}{1} \right|
\end{aligned}
$$

and for $j \geq 0$:

$$B_{N+j} = \left| B_j \tbinom{N}{N} - B_{j+1} \tbinom{N}{N-1} + \ldots \pm B_{j+N-1} \tbinom{N}{1} \right|.$$

These expressions can be simplified:

$$
\begin{aligned}
B_0 &= \binom{N}{0} = 1 \\
B_1 &= \binom{N}{1} - 1 = N - 1 \\
B_2 &= \left| \binom{N}{2} - B_1 \binom{N}{1} \right| = \left| \frac{N(N-1)}{2} - N(N-1) \right| = \frac{N(N-1)}{2} \\
B_3 &= \left| \binom{N}{3} - B_1 \binom{N}{2} + B_2 \binom{N}{1} \right| = \left| \frac{N(N-1)(N-2)}{3!} - (N-1)\frac{N(N-1)}{2} + \frac{N(N-1)}{2} N \right| \\
&= \frac{N(N-1)(N-2)}{3!} + (-N+1+N)\frac{N(N-1)}{2} = \frac{3N(N-1) + N(N-1)(N-2)}{3!} \\
&= \frac{N(N-1)(3+N-2)}{3!} = \frac{(N+1)N(N-1)}{3!} \\
&\vdots
\end{aligned}
$$

We show that for $-N \leq j$, $B_{N+j} = \binom{2N+j-2}{N+j}$ by induction:

Suppose for $M \in \mathbb{N}$, $B_M = \binom{M+N-2}{M} = \frac{(M+N-2)!}{M!(N-2)!}$. The Betti number for $M+1$ is given by:

$$
\begin{aligned}
B_{M+1} &= \left| B_{M+1-N} \binom{N}{N} - B_{M+1-N+1} \binom{N}{N-1} + \ldots \pm B_M \binom{N}{1} \right| \\
&= \left| \left(\binom{M-1}{M+1-N}, -\binom{M}{M+1-N+1}, \ldots, \pm \binom{M+N-2}{M} \right) \cdot \left(\binom{N}{0}, \binom{N}{1}, \ldots, \binom{N}{N-1} \right) \right|.
\end{aligned}
$$

Let $B(x) = \binom{M-1+x}{M+1-N+x} = \frac{1}{(N-2)!}(x+M-1)(x+M-2)\cdots(X+M-N+2)$. Since $B(x)$ is a polynomial with degree $N-2$,

$$(B(x), B(x+1), \ldots, B(x+N)) \cdot \left(\binom{N}{0}, -\binom{N}{1}, \ldots, \pm\binom{N}{N}\right) = 0$$

by Theorem 3.8.1. Let $x = 0$. Then

$$\begin{aligned}
&(B(x), B(x+1), \ldots, B(x+N)) \cdot \left(\binom{N}{0}, -\binom{N}{1}, \ldots, \pm\binom{N}{N}\right) \\
=\ & (B(0), B(1), \ldots, B(N)) \cdot \left(\binom{N}{0}, -\binom{N}{1}, \ldots, \pm\binom{N}{N}\right) \\
=\ & \left| \left(\binom{M-1}{M+1-N}, -\binom{M}{M+1-N+1}, \ldots, \pm\binom{M+N-1}{M+1}\right) \cdot \left(\binom{N}{0}, \binom{N}{1}, \ldots, \binom{N}{N}\right) \right| \\
=\ & 0.
\end{aligned}$$

Hence $B_{M+1} = B(N) = \binom{M+N-1}{M+1}$. □

So for any $x \in \mathbb{N}$,

$$B_x = \binom{x+N-2}{x} = \frac{(x+N-2)!}{x!(N-2)!} = \frac{1}{(N-2)!}(x+1)(x+2)\cdots(x+N-2).$$

Corollary 4.2.2. *For any valise Adinkra with $2k$ vertices, the Betti polynomial is*

$$\frac{k}{(N-2)!}(x+1)(x+2)\cdots(x+N-2).$$

4.3 A Spanning Set for Betti Polynomials

In Section 4.2, we were able to find the Betti polynomials for valise Adinkras with N edge colors by taking the homology of the (1,1) graph with the free Adinkra with N edges. We can take this further to develop a spanning set for all of the polynomials corresponding to Adinkras.

Any Adinkra has N edge colors; a_0 vertices at height 0, a_1 vertices at height 1, etc., with a_N vertices at height N, for $a_0, a_1, \ldots, a_N \in \mathbb{N}$; and s_i sources at height i for $0 < i \leq N-1$, $s_i \in \mathbb{N}$. In this section, we focus our attention to Adinkras for which $s_i = 0$ for $i > 0$, i.e. Adinkras whose only sources are at height 0.

We compute the Betti polynomials for the (1) Adinkra. Note that the $(0,1)$ is the (1) Adinkra, shifted down a row. This means that the first Betti number will be 0, and from then on $b_n(0,1) = b_{n-1}(1)$. That is the sequence is shifted by 1; this has the effect of shifting the polynomial, therefore we can substitute $x = x-1$ into the polynomial for (1) to obtain the polynomial for (0,1). Hence, we need only calculate the polynomials for the Adinkra (1) for $N \geq 2$. This is nearly the same as what we did in Section 4.2; this time b_x is given by

$$B(x) = \frac{1}{(N-1)!}(x+1)(x+2)\cdots(x+N-1) = \binom{x+N-1}{x}.$$

	$N=2$	$N=3$	$N=4$	$\dots$
(1)	$x+1$	$\frac{1}{2}(x+1)(x+2)$	$\frac{1}{3!}(x+1)(x+2)(x+3)$	$\dots$
$(0,1)$	x	$\frac{1}{2}(x)(x+1)$	$\frac{1}{3!}(x)(x+1)(x+2)$	$\dots$
$(0,0,1)$	$x-1$	$\frac{1}{2}(x)(x-1)$	$\frac{1}{3!}(x-1)(x)(x+1)$	$\dots$
$(0,0,0,1)$	$x-2$	$\frac{1}{2}(x-1)(x-2)$	$\frac{1}{3!}(x-2)(x-1)(x)$	$\dots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Using the table, we can find the Betti polynomials for Adinkras with $(a_0, a_1, a_2, \dots, a_N)$, where a_i corresponds to the number of vertices in each row, by evaluating

$$(a_0, a_1, a_2, \dots, a_N) \cdot (B(x), -B(x-1), B(x-2), \dots, \pm B(x-N).$$

If the first coefficient of the polynomial is negative, we multiply by -1.

Example 4.3.1. The Betti polynomial for $(3,4,1)$ Adinkra with 3 edge colors is given by

$$3\left(\frac{1}{2}(x+1)(x+2)\right) - 4\left(\frac{1}{2}(x)(x+1)\right) + 1\left(\frac{1}{2}(x)(x-1)\right) = 2x+3,$$

since

$$(3,4,1) = 3(1,0,0) + 4(0,1,0) + 1(0,0,1).$$

$\lozenge$

Example 4.3.2. Since $(3,8,5) = 3(1,0,0)+8(0,1,0)+5(0,0,1)$, the Betti polynomial for the $(3,8,5)$ Adinkra with 4 edge colors is given by:

$$3\left(\frac{1}{3!}(x+1)(x+2)(x+3)\right)-8\left(\frac{1}{3!}(x)(x+1)(x+2)\right)+5\left(\frac{1}{3!}(x-1)(x)(x+1)\right)$$
$$= -x^2+2x+3.$$

Note that the leading coefficient is negative so we multiply by -1 and obtain the polynomial x^2-2x-3. ♢

To understand certain phenomena such as constant polynomials and degree $N-2$ polynomials, we look at the expansion of these polynomials for $2 \leq N \leq 5$.

	$N=2$	$N=3$	$N=4$	$N=5$
(1)	$x+1$	$\frac{1}{2}(x^2+3x+2)$	$\frac{1}{6}(x^3+6x^2+11x+6)$	$\frac{1}{24}x^4+10x^3+35x^2+50x+24$
$(0,1)$	x	$\frac{1}{2}(x^2+x)$	$\frac{1}{6}(x^3+3x^2+2x)$	$\frac{1}{24}x^4+6x^3+11x^2+6x$
$(0,0,1)$	$x-1$	$\frac{1}{2}(x^2-x)$	$\frac{1}{6}(x^3-x)$	$\frac{1}{24}x^4+2x^3-x^2-2x$
$(0,0,0,1)$		$\frac{1}{2}(x^2-3x+2)$	$\frac{1}{6}(x^3+3x^2+2x)$	$\frac{1}{24}x^4-2x^3-x^2+2x$
$(0,0,0,0,1)$			$\frac{1}{6}(x^3-6x^2+11x-6)$	$\frac{1}{24}x^4-6x^3+11x^2-6x$
$(0,0,0,0,0,1)$				$\frac{1}{24}x^4-10x^3+35x^2-50x+24$

For an Adinkra with N edge colors, the maximal height is N; an Adinkra with height N corresponds to the free Adinkra (recall the top row of an Adinkra has height 0). Computing the Betti polynomial for an Adinkra with sources at heights other than 0 is slightly more complicated, and we postpone the discussion of such polynomials until Section 4.7. For the time being, we look only at Adinkras will all sources in the top row. Therefore all Betti polynomials for an Adinkra with N edge colors are given by

$$p(x) = (a_0,a_1,a_2,\ldots,a_N)\cdot(B(x),-B(x-1),B(x-2),\ldots,\pm B(x-N)).$$

That is, the polynomials are as follows:

$$N=2: \quad \pm((a_0-a_1+a_2)x+(a_0-a_2))$$

$$N=3: \quad \pm\frac{1}{2}\left((a_0-a_1+a_2-a_3)x^2+(3a_0-a_1-a_2+3a_3)x+(2a_0-2a_3)\right)$$

$$N=4: \quad \pm\frac{1}{6}((a_0-a_1+a_2-a_3+a_4)x^3+(6a_0-3a_1+3a_3-6a_4)x^2 + (11a_0-2a_1-a_2-2a_3+11a_4)x+(6a_0-6a_4))$$

$$N=5: \quad \pm\frac{1}{24}((a_0-a_1+a_2-a_3+a_4-a_5)x^4+(10a_0-6a_1+2a_2+2a_3-6a_4+10a_5)x^3 + (35a_0-11a_1-a_2+a_3+11a_4-35a_5)x^2+(50a_0-6a_1-2a_2-2a_3-6a_4+50a_5)x + (24a_0-24a_5))$$

From these equations we see that that for Adinkras whose only sources are at height 0, the degree of the Betti polynomials is always less than or equal to $N-2$. This is interesting because the polynomials can be of degree $N-1$. However, the equations show that the coefficient for the x^{N-1} term is $a_0-a_1+a_2-\ldots\pm a_n=0$. This follows from the fact that the sign is alternating, and in an Adinkra graph, the rows are alternating between rows of black and white vertices. Since there are an equal number of black and white vertices $a_0+a_2+\ldots a_{2k}=a_1+a_3+\ldots a_{2k\pm1}$, hence $(a_0+a_2+\ldots a_2k)-(a_1+a_3+\ldots a_{2k\pm1})=a_0-a_1+a_2-\ldots\pm a_N=0$.

Also, note that if $a_0=a_n$, the polynomial will not have a constant term.

We will return to the construction of polynomials for Adinkras with sources in other rows in Section 4.7.

4.4 Difference Sequences

In this section, and in Section 4.5 we introduce two new topics from Combinatorics that have interesting applications in the context of our research. The first is difference sequences.

Given a sequence of numbers $a_0,a_1,a_2,\ldots,a_n,\ldots$, we can look at the difference between each term in the sequence, and repeat the process, looking at the difference of the differences, and so on.

Definition 4.4.1. Let $n \geq 0$. We denote $a_{n+1} - a_n$ by Δa_n. For $k \geq 1$, the k^{th} **order difference sequence** is

$$\Delta^k a_0,\ \Delta^k a_1,\ \Delta^k a_2,\ \ldots,\ \Delta^k a_n,\ \ldots$$

where $\Delta^k a_n = \Delta(\Delta^{k-1} a_n)$. We define $\Delta^0 a_n = a_n$; that is the 0^{th} order difference sequence is the sequence itself. △

Definition 4.4.2. The **difference table** for a sequence is obtained as follows: For $k = 0, 1, 2, \ldots$, the k^{th} row of the table is given by the k^{th} order difference sequence. △

Example 4.4.3. Let $\{s_n\}_{n=0}^{\infty}$ be the sequence given by $p(n) = n^2 + 2n + 1$. The difference table for this sequence is

$$\begin{array}{c} 1 \quad 4 \quad 9 \quad 16 \quad 25 \quad 36 \ldots \\ 3 \quad 5 \quad 7 \quad 9 \quad 11 \ldots \\ 2 \quad 2 \quad 2 \quad 2 \ldots \\ 0 \quad 0 \quad 0 \ldots \\ \ldots \end{array}$$

with the top row (or 0^{th} row) given by $\Delta^0 a_n = a_n$. Note that since $\Delta^3 a_n = 0$ for all $n \geq 0$, for $k > 3$, $\Delta^k a_n = 0$ for all $n \geq 0$. ♢

Using k-order sequences, we state the following theorem, whose proof can be found in [1].

Theorem 4.4.4. *Let h_n be a term in the sequence given by a polynomial of degree k, i.e.*

$$h_n = a_k n^k + a_{k-1} n^{k-1} + \ldots + a_1 n + a_0, \quad (n \geq 0).$$

Then $\Delta^{k+1} h_n = 0$ for all $n \geq 0$.

The preceding statement follows from the fact that at each step the degree of the polynomial corresponding to the sequence decreases by 1.

Note that in Example 4.4.3, the polynomial has degree 2, and the 3^{rd} row of the table is 0, which is consistent with the theorem.

Example 4.4.5. Let h_n be the sequence given by the polynomial

$$h_n = a_k n^k + a_{k-1} n^{k-1} + \ldots + a_1 n + a_0.$$

Then

$$\begin{aligned}\Delta h_n = & \; a_k (n+1)^k - a_k n^k + a_{k-1}(n+1)^{k-1} + \ldots + a_0 - a_0 \\ = & \; a_k \left(\binom{k}{1} n^{k-1} + \binom{k}{2} n^{k-2} + \ldots + \binom{k}{k} \right) \\ & + \; a_{k-1} \left(\binom{k-1}{1} n^{k-2} + \binom{k-1}{2} n^{k-3} + \ldots + \binom{k-1}{k-1} \right) \\ & + \; \ldots \; + \; a_1.\end{aligned}$$

If we were to compute $\Delta^2 h_n$, we would obtain a polynomial of degree $k-2$, etc. ♢

Therefore we can use the difference tables to determine the degree of the polynomial corresponding to a sequence.

Careful attention shows that this result is equivalent to our Theorem 3.8.1, i.e. our theorem can be stated in terms of k^{th} order difference sequences.

Let $\{a_k\}_{k=0}^{\infty}$ be a sequence. Then

$$\begin{aligned}\Delta\{a_k\} &= \{a_1 - a_0,\; a_2 - a_1,\; a_3 - a_2,\; \ldots\} \\ &= \{(1,-1)\cdot(a_{k+1}, a_k)\}_{k=0}^{\infty} \\ \Delta^2\{a_k\} &= \{(a_2 - a_1) - (a_1 - a_0),\; (a_3 - a_2) - (a_2 - a_1),\; \ldots\} \\ &= \{a_2 - 2a_1 + a_0, a_3 - 2a_2 + a_1, \ldots\} \\ &= \{(1,-2,1)\cdot(a_{k+2}, a_{k+1}, a_k)\}_{k=0}^{\infty}.\end{aligned}$$

It turns out that

$$\Delta^n\{a_k\} = \left\{ \left(\binom{n}{0}, -\binom{n}{1}, \binom{n}{2}, \dots, \pm\binom{n}{n} \right) \cdot (a_{n+k}, a_{n+k-1}, \dots, a_k) \right\}_{k=0}^{\infty}.$$

We prove this by induction, but first, we provide the following well known Lemma, which is the defining relation of Pascal's triangle:

Lemma 4.4.6. *Let $n,k \in \mathbb{N}$. Then $\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}$.*

Theorem 4.4.7. *Let $\{a_k\}_{k=0}^{\infty}$ be a sequence. And let $\Delta^n\{a_k\}$ be the n^{th} order difference sequence. Then*

$$\Delta^n\{a_k\} = \left\{ \left(\binom{n}{0}, -\binom{n}{1}, \binom{n}{2}, \dots, \pm\binom{n}{n} \right) \cdot (a_{n+k}, a_{n+k-1}, \dots, a_k) \right\}_{k=0}^{\infty}.$$

Proof. Our proof is by induction. Earlier work shows that the theorem is true for $n = 1$ and $n = 2$, so suppose the theorem holds for n, i.e.

$$\Delta^n\{a_k\} = \left\{ \left(\binom{n}{0}, -\binom{n}{1}, \binom{n}{2}, \dots, \pm\binom{n}{n} \right) \cdot (a_{n+k}, a_{n+k-1}, \dots, a_k) \right\}_{k=0}^{\infty}.$$

Then,

$$\begin{aligned}
\Delta^{n+1}\{a_k\} &= \left\{ \left(\binom{n}{0}, -\binom{n}{1}, \binom{n}{2}, \dots, \pm\binom{n}{n} \right) \cdot (a_{n+1+k}, a_{n+k}, \dots, a_{k+1}) \right. \\
&\qquad \left. - \left(\binom{n}{0}, -\binom{n}{1}, \binom{n}{2}, \dots, \pm\binom{n}{n} \right) \cdot (a_{n+k}, a_{n+k-1}, \dots, a_k) \right\}_{k=0}^{\infty} \\
&= \left\{ a_{n+k+1} - \left(\binom{n}{1} + \binom{n}{0} \right) a_{n+k} + \left(\binom{n}{2} + \binom{n}{1} \right) a_{n+k-1} - \dots \pm \left(\binom{n}{n} + \binom{n}{n-1} \right) a_{k+1} \pm a_k \right\}_{k=0}^{\infty}.
\end{aligned}$$

Therefore we have

$$\begin{aligned}
\Delta^{n+1}\{a_k\} &= \left\{ a_{n+k+1} - \binom{n+1}{1} a_{n+k} + \binom{n+1}{2} a_{n+k-1} - \dots \pm \binom{n+1}{n} a_{k+1} \pm a_k \right\}_{k=0}^{\infty} \\
&= \left\{ \left(\binom{n+1}{0}, -\binom{n+1}{1}, \binom{n+1}{2}, \dots, \pm\binom{n+1}{n+1} \right) \cdot (a_{n+k+1}, a_{n+k}, \dots, a_k) \right\}_{k=0}^{\infty}
\end{aligned}$$

by Lemma 4.4.6. □

By Theorem 4.4.4, for a sequence $\{a_k\}_{k=0}^{\infty}$, if a_k is given by a degree n polynomial, $p(x)$, then $\Delta^{n+1}a_k = 0$. By our Theorem 3.8.3, if a_k is given by some function $f : \mathbb{N} \cup \{0\} \to \mathbb{N} \cup \{0\}$ and

$(f(x), f(x+1), \ldots, f(x+n)) \cdot \left(\binom{n}{0}, -\binom{n}{1}, \binom{n}{2}, \ldots, \pm\binom{n}{n}\right) = 0$ for all x, then f is a polynomial with degree $\leq n$. Let $\Delta^{n+1} a_k = 0$ for all k. Then

$$(f(x), f(x+1), \ldots, f(x+n)) \cdot \left(\binom{n}{0}, -\binom{n}{1}, \binom{n}{2}, \ldots, \pm\binom{n}{n}\right) = 0,$$

hence a_k is given by a polynomial with degree $\leq n$.

We have just shown that the converse of the statement in Theorem 4.4.4 is also true, i.e. we have the following biconditional:

Theorem 4.4.8. *Let $\{a_k\}_{k=0}^{\infty}$ be a sequence. Then $\Delta^{n+1} a_k = 0 \ \forall \ k \in \mathbb{N} \cup \{0\}$ if and only if a_k is given by a polynomial with degree less than or equal to n.*

4.5 Stirling Numbers

The set $\{1, x, x^2, \ldots, x^n\}$ is a basis for the space of polynomials with degree $\leq n$. Another basis is $\{\binom{x}{0}, \binom{x}{1}, \ldots, \binom{x}{n}\}$, and a third basis is given by

$$\{P(x+n-1, x-1), P(x+n-2, x-1), P(x+n-3, x-1), \ldots, P(x-1, x-1)\},$$

or the "rising factorial basis", which can be expressed in terms of the monomial basis: $\{1, x, x^2, \ldots, x^n\}$.

Definition 4.5.1. The **unsigned Stirling numbers of the first kind**, denoted $|s(n,k)|$, are the coefficients of the rising factorial basis expressed as a polynomial. △

Example 4.5.2. Two bases for the space of polynomials with degree 4 are $\{1, x, x^2, x^3, x^4\}$, and $\{(x+3)(x+2)(x+1)x,\ (x+2)(x+1)x,\ (x+1)x,\ x,\ 1\}$, the rising factorial basis. If we expand the polynomials in the latter basis, we obtain $\{x^4 + 6x^3 + 11x^2 + 6x,\ x^3 + 3x^2 + 2x,\ x^2 + x,\ x,\ 1\}$. If we express this basis in terms of the first basis, we have

$$\{(0,6,11,6,1),\ (0,2,3,1),\ (0,1,1),\ (0,1),\ (1)\} = \bigcup_{n=0}^{4} \left(\bigcup_{k=0}^{n} |s(n,k)| \right).$$

◊

Yet another basis is the "falling factorial basis", given by

$$\{x,\ x(x-1),\ x(x-1)(x-2),\ \ldots,\ x(x-1)\cdots(x-n+1)\},$$

or

$$\{P(x,1),\ P(x,2),\ \cdots,\ P(x,n)\}.$$

Definition 4.5.3. The **signed Stirling numbers of the first kind**, denoted $s(n,k)$, are the coefficients of the falling factorial basis expressed as a polynomial. △

The Stirling numbers are generated from the following recurrence relation:

$$s(n,0) = 0, \qquad (n \geq 1)$$

and

$$s(n,n) = 1, \qquad (n \geq 0)$$

If $1 \leq k \leq n-1$, then

$$s(n,k) = (n-1)s(n-1,k) + s(n-1,k-1).$$

For $0 \leq n,k < 10$, the Stirling numbers of the first kind are given in the following table.

$n\backslash k$	0	1	2	3	4	5	6	7	8	9
0	1									
1	0	1								
2	0	−1	1							
3	0	2	−3	1						
4	0	−6	11	−6	1					
5	0	24	−50	35	−10	1				
6	0	−120	274	−225	85	−15	1			
7	0	720	−1764	1624	−735	175	−21	1		
8	0	−5040	13068	−13132	6769	−1960	322	−28	1	
9	0	40320	−109584	118124	−67284	22449	−4536	546	−36	1

An entry in row n and column k, $(n,\ k)$ is obtained by $(n-1)\cdot(n-1,\ k)+(n-1,\ k-1)$, i.e. by multiplying the absolute value of the entry above by its row number and adding the absolute value of the entry to the left of that entry. The sign is given by $(-1)^{n-k}$. The unsigned Stirling numbers have the same absolute value.

4.6 An Equation for Betti Polynomials

In Section 4.3, we showed that the Betti Polynomials are given by

$$b(x) = (a_0,\ a_1,\ \ldots,\ a_N)\cdot(B(x),\ -B(x-1),\ \ldots,\ \pm B(x-N)),$$

where $B(x) = \frac{1}{(N-1)!}(x+1)(x+2)\cdots(x+N-1)$. From Section 4.4, we have

$$\sum_{k=0}^{N-j} |s(j,k)|\, x^k = x(x+1)(x+2)\cdots(x+(N-j)-1),$$

and

$$\sum_{k=0}^{j} s(j,k)\, x^k = x(x-1)(x-2)\cdots(x-j+1),$$

hence

$$\sum_{k=1}^{j} s(j,k)\, x^{k-1} = (x-1)(x-2)\cdots(x-j+1).$$

Therefore

$$\begin{aligned} B(x-j) &= \frac{1}{(N-1)!}(x-j+1)\cdots(x-1)(x)(x+1)\cdots(x+N-j-1) \\ &= \frac{1}{(N-1)!}\sum_{k=1}^{j} s(j,k)\, x^{k-1}\sum_{k=0}^{N-j} |s(j,k)|\, x^k. \end{aligned}$$

Theorem 4.6.1. *Let A be an Adinkra with a_j vertices at height j for $j \in \{0,1,\ldots,N\}$, and $s_j = 0$ for $i > 0$. Then the Betti polynomial for A can be expressed in terms of Stirling numbers, and coefficients a_j as :*

$$b_A(x) = \pm\frac{1}{(N-1)!}\sum_{j=0}^{N}\left(a_j(-1)^j\sum_{k=0}^{N-1}\left(\sum_{i=0}^{k} s(j,i+1)\,|s(N-j,k-i)|\right)x^k\right).$$

4.7 An Expression for the Betti Polynomial of any Adinkra

Let $(c_0,\ c_{1_{k_1}}^{s_1},\ c_{2_{k_2}}^{s_2},\ \ldots,\ c_{n-2_{k_{n-2}}}^{s_{n-2}},\ c_{n-1},\ c_n)$ be an Adinkra with c_i vertices at height i, s_i sources at height i, and k_i sinks at height i, for $i \in \mathbb{N} \cup \{0\}$. Recall, this is a general form for all Adinkras, as $s_0 = c_0$, and an Adinkra cannot have sources at height N or $N-1$. The only Adinkra with $c_i \neq 0\ \forall\ i \in \{0,1,\ldots,N\}$ is the free Adinkra.

We calculate the Betti numbers for an Adinkra as follows:

$$b_0 = (c_0,\ s_1,\ s_2,\ \ldots,\ s_{n-2}),$$

For $j \geq 0$, we let

$$a_j = -c_j + \sum_{k=0}^{j} s_k \binom{N}{j-k},$$

Then for $x > 0$, b_x is given by:

$$b(x) = \frac{1}{(N-1)!} \sum_{j=0}^{2N-1} \left(a_j(-1)^j \sum_{k=0}^{2N-2} \left(\sum_{i=0}^{k} s(j,i+1)\,|s(N-j,k-i)| \right) x^k \right).$$

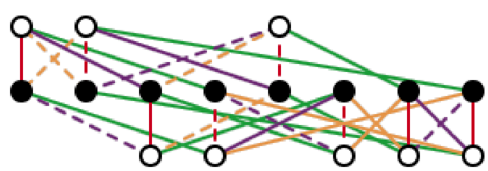


Figure 4.7.1. The $(3, 8^1, 5)$ Adinkra with $N = 4$

Example 4.7.1. We calculate the Betti numbers for the $(3, 8^1, 5)$ Adinkra with 4 edge colors, shown in Figure 4.7.1. There are 3 sources at height 0, and there is 1 source at height 1, so $b_0 = (3, 1)$.

To obtain the Betti numbers, b_x for $x > 0$, we calculate coefficients a_j in Figures 4.7.2 and 4.7.3. The coefficients a_j are $(0,\ 5,\ 17,\ 18,\ 7,\ 1)$.

$$3\begin{pmatrix}1\\4\\6\\4\\1\end{pmatrix}+\begin{pmatrix}1\\4\\6\\4\\1\end{pmatrix}=\begin{pmatrix}3\\13\\22\\18\\7\\1\end{pmatrix}$$

Figure 4.7.2. For each source, s_i, we add s_i copies of the free Adinkra

$$\begin{pmatrix}3\\13\\22\\18\\7\\1\end{pmatrix}-\begin{pmatrix}3\\8\\5\end{pmatrix}=\begin{pmatrix}0\\5\\17\\18\\7\\1\end{pmatrix}$$

Figure 4.7.3. We subtract the number of vertices in the original Adinkra

We then compute the homology on the vector we are left with; the Betti numbers b_x, $x > 0$ for the $(3, 8^1, 5)$ Adinkra are given by the Betti numbers, b_x of the $(0,\ 5,\ 17,\ 18,\ 7,\ 1)$. ♢

If we take the homology of the Adinkra in Example 4.7.1 one step further we have:

$$\begin{pmatrix}0\\5\\17\\18\\7\\1\end{pmatrix}-5\begin{pmatrix}1\\4\\6\\4\\1\end{pmatrix}=\begin{pmatrix}0\\0\\-3\\-12\\-13\\-4\end{pmatrix}$$

We can compare this to the homology of the $(3, 8, 5)$ Adinkra given in the image below.

$$3\begin{pmatrix}1\\4\\6\\4\\1\end{pmatrix}=\begin{pmatrix}3\\12\\18\\12\\3\end{pmatrix}-\begin{pmatrix}3\\8\\5\end{pmatrix}=\begin{pmatrix}0\\4\\13\\12\\3\end{pmatrix}-4\begin{pmatrix}1\\4\\6\\4\\1\end{pmatrix}=\begin{pmatrix}0\\0\\-3\\-12\\-13\\-4\end{pmatrix}$$

We see that by adding 3 copies of the free Adinkra and then subtracting 4, we obtain the same result as we do by adding $(3, 1)$ and then -5 copies of the free Adinkra. Therefore the polynomials for the $(3, 8, 5)$ and $(3, 8^1, 5)$ are equal, however their first two Betti numbers differ.

In this case the polynomials are equal, however, as mentioned in Section 4.1, the polynomials corresponding to the $(2, 8, 6)$ and $(2, 8^2, 6)$ differ: the associated polynomials are $2x^2 + 4x$ and

$2x^2 - 2$ respectively. Although the polynomials are different, note that if we plug $x-1$ into $2x^2 + 4x$, we obtain $2x^2 - 2$. So the sequence of numbers is the same, it is only shifted, as can be seen in Appendix B.

We believe that this will always be the case:

Conjecture 4.7.2. *Let A and B be two Adinkras with N edge colors such that the number of vertices at each height agree. Let $p(x)$ be the polynomial corresponding to the Betti numbers for A, and $q(x)$ be the polynomial corresponding to B. Then for some k,* $0 \leq k < N, p(x) = q(x \pm k)$.

The conjecture implies that the polynomials will have the same degree, hence we can extend our result from Section 4.3 to Adinkras with sources at non-zero heights.

Conjecture 4.7.3. *Let A be an Adinkra with N edge colors, then the polynomial corresponding to the Betti numbers of A has degree* $\leq N - 2$.

4.8 Vertex Lowering

As pointed out in [3], although the vertex lowering operation only affects the height of the vertex being lowered, it may result in additional sources. For example, in Figure 4.8.1, lowering the vertex at height 0, results in 3 new sources.

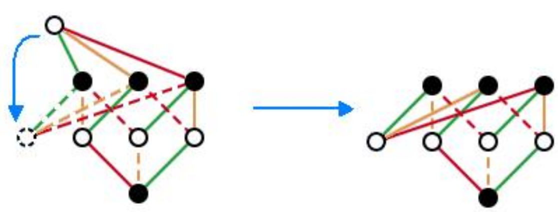


Figure 4.8.1. A vertex lowering resulting in new sources

In many cases, lowering a vertex affects only the vertex that is lowered. In these cases, the polynomials of the Adinkra before and after the vertex lowering should differ by the polynomial corresponding to the homology of $(-1, 0, 1)$. This follows from the fact that by performing the vertex lowering operation on a vertex in a given row, the number of vertices in that row decreases by 1, and the number of vertices two rows below is increased by 1.

We find that for $N = 2$, this polynomial is equal to 2, and for $N > 2$ the polynomial corresponding to $(-1,0,1)$, is given by

$$L(N) = \frac{2}{(N-2)!}\left(x + \frac{N-2}{2}\right)(x+1)(x+2)\cdots(x+N-3).$$

In the table below we give $L(N)$ for $2 \le N \le 5$:

	$N=2$	$N=3$	$N=4$	$N=5$
(-1,0,1)	2	$2(x+\frac{1}{2})$	$(x+1)^2$	$\frac{1}{3}(x+\frac{3}{2})(x+1)(x+2)$

If we perform a vertex lowering operation on a vertex with height $h \neq 0$, we substitute $(x-h)$ into the equation $L(N)$. The polynomial $L(N)$ is the difference between the Betti polynomials corresponding to the Adinkra before and after the vertex lowering. And,

$$L(N,j) = \left(\frac{1}{(N-1)!}\sum_{k=0}^{N-3}|s(j,k+1)| + \frac{2}{(N-2)!}\sum_{k=0}^{N-2}|s(j,k)|\right)x^k.$$

Usually, adding or subtracting the $(-1,0,1)$ polynomial has the desired effect.

Example 4.8.1. The $(6,8,2)$ Adinkra with $N = 4$, has the Betti polynomial

$$2(x+1)(x+3) = 2x^2 + 8x + 6.$$

From the table, we have:

$$L(4,0) = (x+1)^2 = x^2 + 2x + 1.$$

By lowering the vertex at height $j = 0$, we obtain the $(5,8,3)$ Adinkra with polynomial $(x+1)(x+5) = x^2 + 6x + 5$. Note that $(2x^2 + 8x + 6) - \left(x^2 + 2x + 1\right) = x^2 + 6x + 5$. ♢

Example 4.8.2. Consider lowering the vertex at height 1 for the Adinkra $(1,5^1,7,3)$, with 4 edge colors, giving the Adinkra $(1,4,7^1,4)$. The $(1,5^1,7,3)$ corresponds to the polynomial $2x+1$, and lowering a vertex at height 1 on an Adinkra with 4 edges corresponds to $L(4,1) = x^2$. The polynomial for the $(1,4,7^1,4)$ is $x^2 + 2x + 1$, which we obtain by adding $L(4,1) = x^2$. ♢

However, there are some cases where even though lowering a vertex only effects that vertex, the difference between the polynomials before and after the operation does not equal $L(N, j)$.

Example 4.8.3. The polynomial corresponding to the homology of the $(4,8,4)$ Adinkra with 4 edge colors is $4x+4$. If we lower a source at height 0 we obtain the $(3,8,5)$ Adinkra with polynomial x^2-4. The difference between the polynomials is x^2+4x+8 which is not equal to $L(4,0)$. ♢

This issue is addressed in the next section in which we discuss future research. Fortunately, there is another way to understand the vertex lowering operation.

By Theorem 2.2.4, all Adinkras with the same topology, i.e. all connected Adinkras with N edges, can be obtained by performing the vertex lowering operation. Up to isomorphism, there are two Adinkras with $N=2$ edge colors, five Adinkras with $N=3$ edge colors, and twenty-two Adinkras with $N=4$ edge colors. For example, we can obtain the $(3,4,1)$ with 3 edge colors by performing a vertex lowering on the $(1,3,3,1)$ Adinkra, or the $(4,4)$ Adinkra.

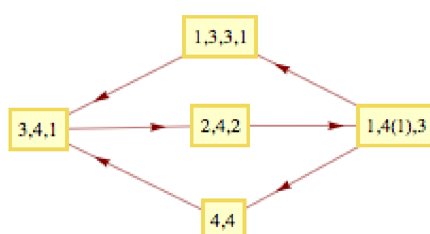


Figure 4.8.2. Vertex lowering graph for Adinkras with 3 edge colors

The graphs in Figures 4.8.2 and 4.8.3 relate the Adinkras with 3 and 4 edge colors respectively by the vertex lowering operation. Each vertex in the figure corresponds to a unique Adinkra with the same topology, and each edge corresponds to a vertex lowering. Therefore vertices are adjacent only if one Adinkra can be obtained by performing a vertex lowering operation on the other.

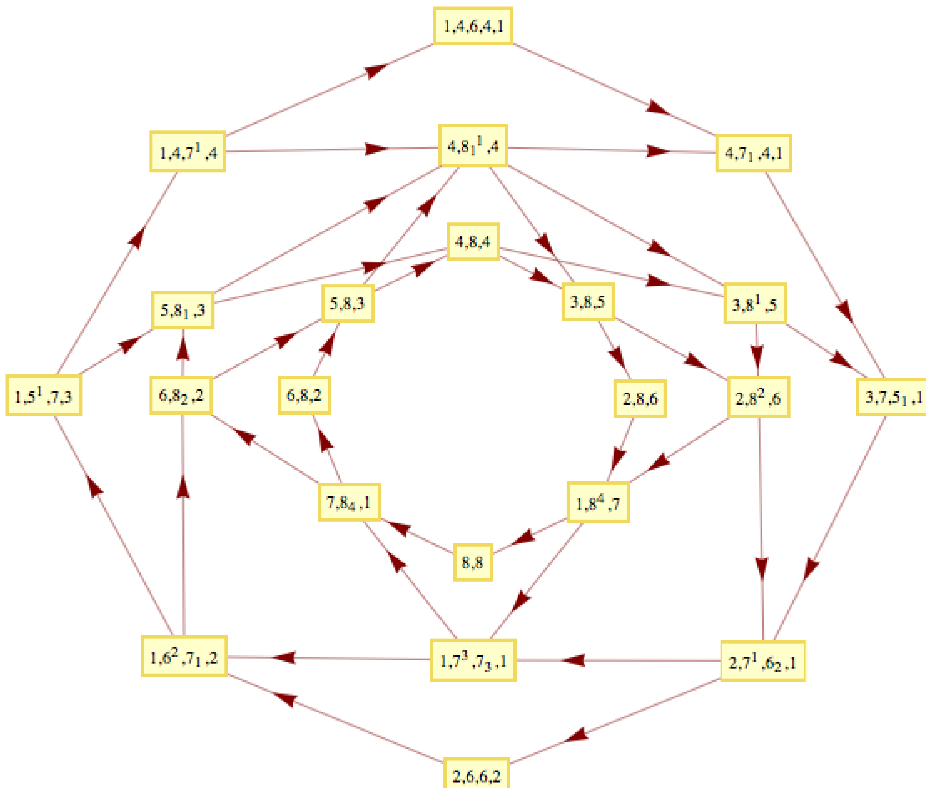


Figure 4.8.3. Vertex lowering graph for Adinkras with 4 edge colors

Both figures possess symmetrical properties, however we examine Figure 4.8.3 in detail, as it is more intricate, and therefore more interesting. A horizontal symmetry is immediately apparent. Careful attention shows that each Adinkra A, has a mirror image in the diagram that corresponds to that Adinkra upside down, which we denote A'; if $A = A'$, the Adinkra is in the center of the figure. All of the edges on the right side of the graph are directed downwards, and all of the edges on the left are directed up, so that if we follow the edges on the right down, it corresponds to a vertex lowering, and if we follow the vertices on the left down, it corresponds to a vertex raising. Also, for any vertex labeled by A, such that $A \neq A'$, the direction of each edge incident to A is reversed for A'. Consequently, if we follow a path from one of the Adinkras in the center of the figure to any other Adinkra A in the graph, following a mirror image of that path from the same initial point gives the Adinkra A'. These properties can also be observed in Figure 4.8.2.

We have not generated a vertex lowering graph for $N > 4$, although we expect it should have the same properties. Understanding these diagrams may lead to an intuition for generating unique Adinkras sharing a topology, or finding a count. Topics such as these will likely be developed further in future research, which we discuss in the following chapter.

5
Future Research

In this paper we discussed several new results including polynomials corresponding to the Betti numbers of an Adinkra and a means by which to compute these polynomials. In examining these polynomials, new and interesting questions arose, which will likely serve as a direction for future research, and if answered, will strengthen our results.

Although alternating signs come up a lot when working with exterior algebras, it is still unclear why we multiply the coefficients of the Adinkra by negative and positive elements of the spanning set. Recall the Betti polynomial for an Adinkra $(a_0, a_1, a_2, \ldots, a_N)$, with N edge colors, is given by

$$(a_0,\ a_1,\ a_2,\ \ldots,\ a_N) \cdot (B(x),\ -B(x-1),\ B(x-2),\ \ldots,\ \pm B(x-N).$$

Also, we do not understand why in some cases we multiply the above expression by -1.

It is interesting that in some cases Adinkras with the same number of edge colors and vertices at each height correspond to the same polynomial and in others they do not. Meanwhile, it does seem as though, in any case, the two have the same sequence. This may also be the case for the sequence corresponding to the cohomology. For example, for the $(3,8,5)$ Adinkra with 4 edge colors, H_2 is the $(5,8,3)$ Adinkra shifted down by 1 so the polynomial corresponding to the $(5,8,3)$,

$$x^2+6x+5 = (x+1)(x+5) = (x-2+3)(x+2+3) = p(x+3),$$

where $p(x) = x^2 - 4$, the polynomial corresponding to $(3,8,5)$. Comparing the polynomials corresponding to the homology and cohomology of an Adinkra, we see that in all of our examples, the two sequences have the same numbers. This relation may be analogous to the Poincaré duality. If this result is consistent for all N, it may imply that the number of sources, sinks, and vertices at each height are sufficient to classify Adinkras.

Understanding the polynomials corresponding to the vertex lowering operation may in fact be very useful, because it could allow us to generate all Betti polynomials from the 0 polynomial. Although, this may prove difficult as we have yet to determine why the vertex lowering polynomial does not act as expected in some cases.

Finally, by developing the vertex lowering graphs introduced in the previous chapter for higher N, we may be able to generate a way to compute the number of distinct Adinkras of a given topology.

Appendix A

Number of vertices for an Adinkra with N edge colors

The number of vertices of an Adinkra depends on the number of edge colors, N. For $N > 3$, there are at least two possibilities for the number of vertices. For $1 \leq N \leq 8$, the number of vertices can be found in the table below.

Number of edge colors	Number of vertices
1	2
2	4
3	8
4	8
4	16
5	16
5	32
6	16
6	32
6	64
7	16
7	32
7	64
7	128
8	16
8	32
8	64
8	128
8	256

Appendix B

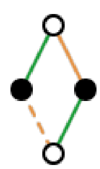
The Betti polynomials for all Adinkras with $N \leq 4$

For $N = 2, 3, 4$ we give all possible Adinkras, up to isomorphism, and the corresponding polynomials.

$N = 2$

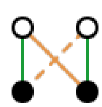
Adinkra: (1,2,1)
Polynomial: 0

Betti numbers: $0, 0, 0, 0, 0, \ldots$



Adinkra: (2,2)
Polynomial: 2

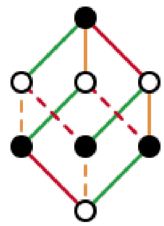
Betti numbers: $2, 2, 2, 2, 2, \ldots$



$N = 3$

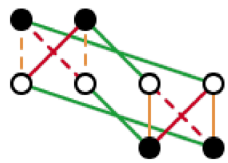
Adinkra: (1,3,3,1)
Polynomial: 0

Betti numbers: $0,0,0,0,0,\ldots$



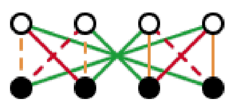
Adinkra: (2,4,2)
Polynomial: 2

Betti numbers: $2,2,2,2,2,\ldots$



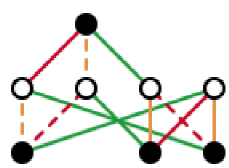
Adinkra: (4,4)
Polynomial: 4x+4
Factorization: 4(x+1)

Betti numbers: $4,8,12,16,20,24,28,32,$
$36,40,44,48,52,56,60,\ldots$



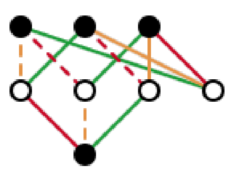
Adinkra: $(1,4^1,3)$
Polynomial: 2x+1

Betti numbers: $(1,1),3,5,7,9,11,13,15$
$17,19,21,23,25,27,29\ldots$



Adinkra: $(3,4_1,1)$
Polynomial: 2x + 3

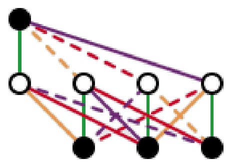
Betti numbers: $3,5,7,9,11,13,15,17$
$19,21,23,25,27,29,31\ldots$



$N = 4$

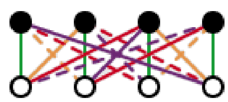
Adinkra: $(1,4,3)$
Polynomial: $x^2 + 2x$
Factorization: $x(x+2)$

Betti numbers: $1, 3, 8, 15, 24, 35, 48, 63, 80, 99, 120$
$143, 168, 195, 224, 255, 288 \ldots$



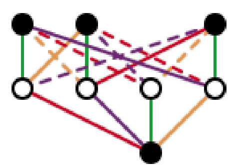
Adinkra: $(4,4)$
Polynomial: $2x^2 + 6x + 4$
Factorization: $2(x+1)(x+2)$

Betti numbers: $4, 12, 24, 40, 60, 84, 112, 144, 180,$
$220, 264, 312, 364, 420, 480, \ldots$



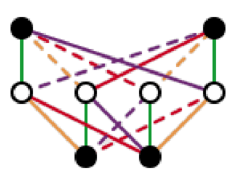
Adinkra: $(3,4,1)$
Polynomial: $x^2 + 4x + 3$
Factorization: $(x+1)(x+3)$

Betti numbers: $3, 8, 15, 24, 35, 48, 63, 80, 99,$
$120, 143, 168, 195, 224, 255 \ldots$



Adinkra: $(2,4,2)$
Polynomial: $2x + 2$
Factorization: $2(x+1)$

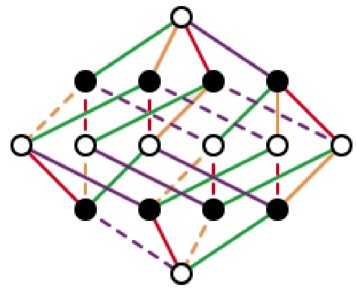
Betti numbers: $2, 4, 6, 8, 10, 12, 14, 16, 18,$
$20, 22, 24, 26, 28, 30, \ldots$



$N = 4$

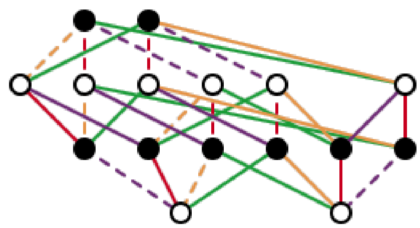
Adinkra: $(1,4,6,4,1)$
Polynomial: 0

Betti numbers: $0,0,0,0,0,\ldots$



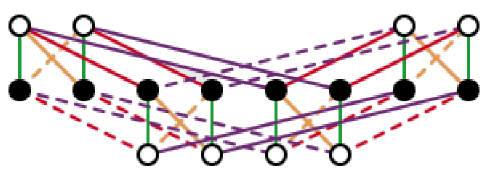
Adinkra: $(2,6,6,2)$
Polynomial: 2

Betti numbers: $2,2,2,2,2,\ldots$



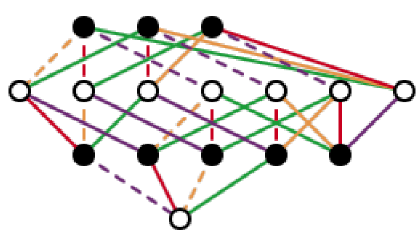
Adinkra: $(4,8,4)$
Polynomial: 4x+4
Factorization: 4(x+1)

Betti numbers: $4,8,12,16,20,24,28,32,$
$36,40,44,48,52,56,60,\ldots$



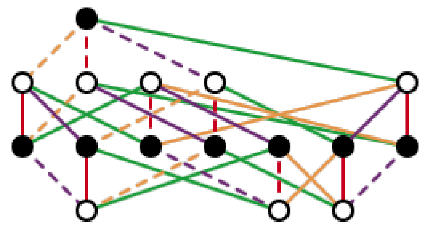
Adinkra: $(3,7,5_1,1)$
Polynomial: $2x+3$

Betti numbers: $3,5,7,9,11,13,15,17,19,$
$21,23,25,27,29,31,\ldots$



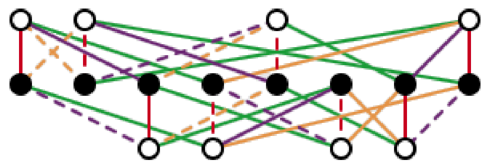
Adinkra: $(1, 5^1, 7, 3)$
Polynomial:$2x+1$

Betti numbers: $(1,1), 3, 5, 7, 9, 11, 13, 15, 17$
$19, 21, 23, 25, 27, 29\ldots$



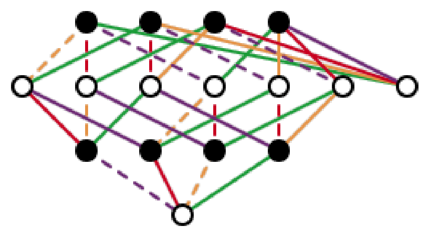
Adinkra: $(4, 8^1_1, 4)$
Polynomial: $4x+4$
Factorization: $4(x+1)$

Betti numbers: $(4,1), 9, 12, 16, 20, 24, 28, 32,$
$36, 40, 44, 48, 52, 56, 60, \ldots$



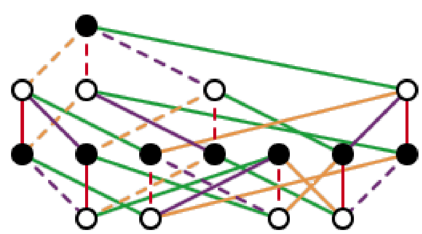
Adinkra: $(4, 7_1, 4, 1)$
Polynomial: x^2+4x+4
Factorization: $(x+2)^2$

Betti numbers: $4, 9, 16, 25, 36, 49, 64, 81, 100,$
$121, 144, 169, 196, 225, 256, \ldots$



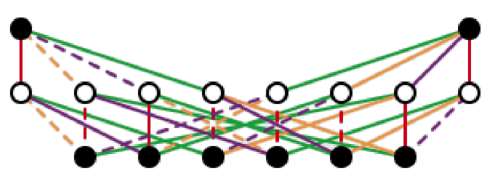
Adinkra: $(1, 4, 7^1, 4)$
Polynomial: x^2+2x+1
Factorization: $(x+1)^2$

Betti numbers: $(1,0,1), 4, 9, 16, 25, 36, 49, 64$
$81, 100, 121, 144, 169, 196, 225$
$196, 225, 256, \ldots$



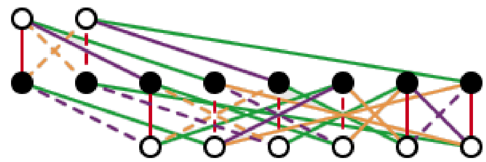
Adinkra: (2,8,6)
Polynomial: $2x^2+4x$
Factorization: $2x(x+2)$

Betti numbers: $2, 6, 16, 30, 48, 70, 96, 160$
$198, 240, 286, 336, 390,$
$448, 510\ldots$



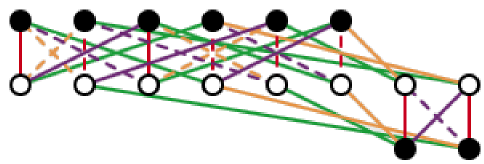
Adinkra: $(2, 8^2, 6)$
Polynomial: $2x^2 - 2$
Factorization: $2(x-1)(x+1)$

Betti numbers: $(2,2), 2, 6, 16, 30, 48, 70, 96, 160, 198, 240, 286, 336, 390, \ldots$



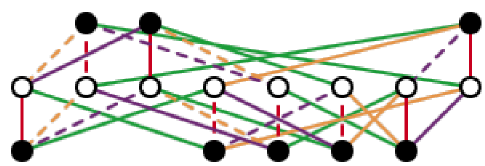
Adinkra: (6,8,2)
Polynomial: $2x^2 + 8x + 6$
Factorization: $2(x+1)(x+3)$

Betti numbers: $6, 16, 30, 48, 70, 96, 160, 198, 240, 286, 336, 390, 448, 510\ldots$



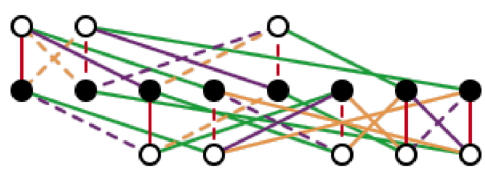
Adinkra: (3,8,5)
Polynomial: $x^2 - 4$
Factorization: $(x-2)(x+2)$

Betti numbers: $3, 4, 3, 5, 12, 21, 32, 45, 60, 77$ $96, 117, 140, 165, 192\ldots$



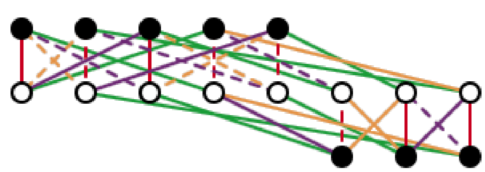
Adinkra: $(3, 8^1, 5)$
Polynomial: $x^2 - 4$
Factorization: $(x-2)(x+2)$

Betti numbers: $(3,1), 5, 3, 5, 12, 21, 32, 45, 60$ $77, 96, 117, 140, 165, 192\ldots$



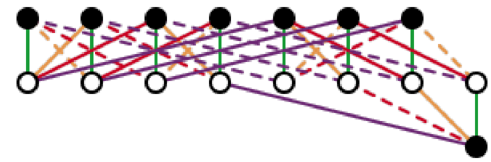
Adinkra: (5,8,3)
Polynomial: $x^2 + 6x + 5$
Factorization: $(x+1)(x+5)$

Betti numbers: $5, 12, 21, 32, 45, 60, 77, 96, 117, 140, 165, 192, 221, 252, \ldots$



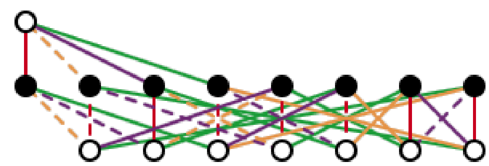
Adinkra: $(7, 8_4, 1)$
Polynomial: $3x^2 + 10x + 7$
Factorization: $3(x+1)(x+\frac{7}{3})$

Betti numbers: 7, 20, 39, 64, 95, 132, 175, 224, 279, 340, 407, 480, 559, 644, 735, ...



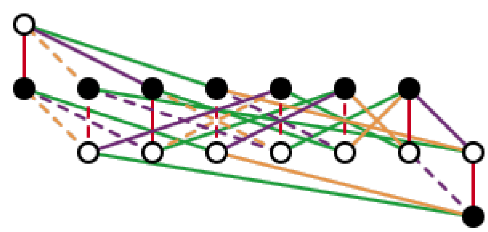
Adinkra: $(1, 8^4, 7)$
Polynomial: $3x^2 + 8x + 4$
Factorization: $3(x+2)(x+\frac{2}{3})$

Betti numbers: (1,4), 15, 32, 55, 84, 119, 160, 160, 207, 260, 319, 384, 455, 532, 615, ...



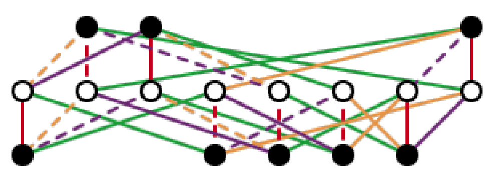
Adinkra: $(1, 7^3, 7_3, 1)$
Polynomial: $2x^2 + 6x + 3$
Factorization: $2(x^2 + 3x + \frac{3}{2})$

Betti numbers: (1,3), 11, 23, 39, 59, 83, 111 143, 179, 219, 263, 311, 363, 419, 479...



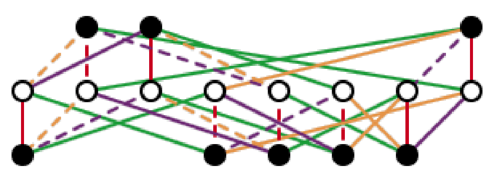
Adinkra: $(1, 6^2, 7_1, 2)$
Polynomial: $x^2 + 4x + 2$

Betti numbers: (1,2), 7, 14, 23, 34, 47, 62, 79, 98, 119, 142, 167, 194, 223, ...



Adinkra: $(2, 7^1, 6_2, 1)$
Polynomial: $x^2 - 2$

Betti numbers: (2,1), 2, 2, 7, 14, 23, 34, 47, 62, 79, 98, 119, 142, 167, 194, ...



Appendix C

Computing the Betti polynomial in Sage

To compute the Betti Polynomial and its factorization for a given Adinkra, the following code can be used in Python, or Sage.

```
def BettiPolynomial(N, Adinkra, Sources=[]):
  ''''
Returns the polynomial corresponding to the Betti numbers of a given Adinkra
graph, and its factorization. The result depends on the number of edge colors,
the number of vertices in each row, and the number of sources in each row of
the graph.

INPUT:
  N -- an integer: the number of edge colors in the Adinkra
  Adinkra -- a list of integers: the number of vertices in each row from
             top to bottom
  Sources -- a list of integers(default []): the number of sources in
             each row from top to bottom

OUTPUT:
  polynomial and factorization of polynomial

EXAMPLES:
  sage: BettiPolynomial(4,[1,8,7],[1,4])
  3*x^2 + 2*x - 1
  (3)*(x - 1/3)*(x + 1)

  sage: BettiPolynomial(4,[3,8,5])
  x^2 - 2*x - 3
  (x - 3)*(x + 1)
```

```
sage: BettiPolynomial(3,[1,3,3,1])
0

''''

print Adinkra, Sources

s=0
if Sources == []:
  Sources = [Adinkra[0]]

if len(Sources) > 1:
    s = 1
    i = 0

    Sc = vector([0]*(len(Sources)))
    for j in [0..len(Sources)-1]:
       Sc[j]+= Sources[j]

    while len(Sources) < 2*N+1:
        Sources.append(0)
        i = i+1
    NL =[]
    j = 0
    while j < 2*N+1:
        k = 0
        next = 0
        while k <= j:
            next += Sources[k]*binomial(N, j-k)
            k = k +1
        NL.append(next)
        j = j +1
    m = 0
    while m < len(Adinkra):
        NL[m]= NL[m] - Adinkra[m]
        m = m +1
    n = 0
    while n < len(NL)-s:
        NL[n] = NL[n+s]
        n = n +1
    Adinkra = NL

    L = len(Adinkra)
    BettiList = [Adinkra[0]]
    i = 1
    while i < 20:
        sumb = 0
        j = 0
        while j < i:
            sumb += BettiList[j]*binomial(N, i-j)
```

```
        j = j+1
    if i < L:
        bi = Adinkra[i]-sumb
    else:
        bi = -sumb
    BettiList.append(bi)
    i = i+1
if s == 1:
    BettiList = [-1*BettiList[i] for i in [0..len(BettiList)-1]]
    BettiList[0] = Sc
BettiListN = [x for x in BettiList if x != 0]
BettiXY = [(i,abs(BettiListN[i])) for i in [L..len(BettiListN)-1]]
print BettiListN
polynomial = PolynomialRing(QQ,'x').lagrange_polynomial(BettiXY)
show(polynomial)
if polynomial != 0:
    print factor(polynomial)
```

Bibliography

[1] Richard Brualdi, *Introductory Combinatorics*, Pearson Prentice Hall, Upper Saddle River, NJ, 2004.

[2] Henri Cartan and Samuel Eilenberg, *Homological Algebra*, Princeton University Press, Chichester, West Sussex, 1999.

[3] C. F. Doran, M. G. Faux, S. J. Gates Jr., T. Hübsch, K. M. Iga, and G. D. Landweber, *On Graph-Theoretic Identifications of Adinkras, Supersymmetry Representations and Superfields*, International Journal of Modern Physics A **22** (2007), 869–929.

[4] ______, *Counter-Examples to a Putative Classification of 1-Dimensional, N-extended Supermultiplets*, Advanced Studies in Theoretical Physics **2** (2008), 100–111.

[5] M. Faux and S. J. Gates, *Adinkras: A Graph Technology for Supersymmetric Representation Theory*, University of Maryland, Center for String and Particle Theory (2004).

[6] K. M. Iga and G. D. Landweber, Private Communication (2008).

[7] Z. Kuznetsova, M. Rojas, and F. Toppan, *Classification of irreps and invariants of the N-extended supersymmetric quantum mechanics*, JHEP **3** (2006), 98.

[8] G. D. Landweber, Private Communication (2009).

[9] Lisa Randall, *Warped Passages*, HarperCollins, New York, NY, 2005.

[10] Steven Roman, *Advanced Linear Algebra*, Springer, Irvine, CA, 2008.

[11] F. Toppan, *Irreps and off-shell invariant actions of the N-extended supersymmetric quantum mechanics*, International Journal of Modern Physics A (2006).